Tutorial 8

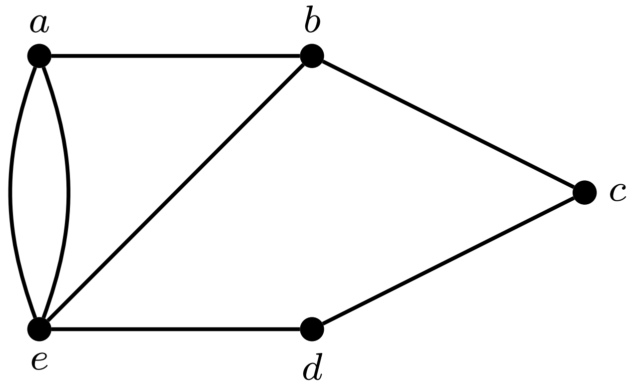
Eulerian tours

Question 1. Let G be a graph with vertex set $V(G) = \{a, b, c, d, e\}$ and edge set $E(G) = \{ab, ae, bc, cd, de, ea, eb\}$.

1. Draw G.
2. Is G connected?
3. Is G simple?
4. List the degrees of every vertex.
5. Find all edges incident to b.
6. List all the neighbors of a.
7. Find a walk, trail, and path in G, each of which has length 3.
8. Find a closed walk, circuit, and cycle in G, each of which starts at e.
9. Is G Eulerian, semi-Eulerian, or neither? Explain your answer.

Solution.

1. One graph representation of G is shown below



2. G is connected
3. G is not simple, there are two edges between a and e
4. $\deg(a) = 3$, $\deg(b) = 3$, $\deg(c) = 2$, $\deg(d) = 2$, $\deg(e) = 4$
5. All edges incident to b include: $ba,\ bc,\ be$.
6. All neighbors of a include: $b,\ e$
7. Example of a walk with length 3: $a,\ b,\ a,\ e$. Example of a trail with length 3: $a,\ e,\ a,\ b$ Example of a path with length 3: $a,\ e,\ d,\ c$

8. Example of a closed walk starting at e: e, a, e, b, a, e. Example of a closed circuit starting at e: e, b, a, e. Example of a closed cycle starting at e: e, a, e

9. Since there are exactly two odd vertices (a, b), G is semi-Eulerian.

Question 2. Which of the following scenarios could be modeled using an Eulerian circuit? Explain your answer.

1. A photographer wishes to visit each of the seven bridges in a city, take photos, then return to his hotel.

2. Salem Public Works must repave all the streets in the downtown area.

3. Frank's Flowers needs to deliver bouquets to 6 customers throughout the city, starting and ending at the flower shop.

4. Richmond Water Authority must read all the water meters throughout the town. One worker is tasked with this job.

5. Sam works in sales for a Fortune 500 company. He spends each day visiting his clients around southwest Virginia and must plan his route to avoid backtracking as much as possible.

Solution.

1. This scenario can be modeled using an Eulerian circuit, as it is reasonable to assume the photographer aims to visit each bridge exactly once and return to the starting location. In the graph representation, the bridges correspond to edges, while the landmasses act as vertices. The existence of an Eulerian circuit ensures that the photographer can traverse every bridge once and return to the starting point (hotel).

2. In this scenario, workers must traverse each street exactly once while repaving, making it a problem related to the Eulerian circuit.

3. This scenario involves visiting specific locations (vertices) rather than covering all streets (edges). It could not be modeled using an Eulerian circuit.

4. Since water meters are likely located at vertices rather than along edges, this problem is not an Eulerian circuit problem.

5. The main goal of Sam is to cover the roads without backtracking, not just visiting clients, thus the scenario can be modeled with the Eulerian circuit.

Question 3. Recall the following theorem from the lecture

Theorem 1 *A graph G is Eulerian if and only if*

- *G is connected and*
- *every vertex has an even degree*

A graph G is semi-Eulerian if and only if

- *G is connected and*

- *exactly two vertices have odd degree*

Explain why a graph cannot be both Eulerian and semi-Eulerian.

Solution. For a graph to be Eulerian, all of its vertices must have even degrees. However, for a graph to be semi-Eulerian, it must contain exactly two vertices of odd degree. These two conditions are mutually exclusive because a single graph cannot simultaneously have all vertices of even degree while also having exactly two vertices of odd degree.

Therefore, a graph cannot be both Eulerian and semi-Eulerian.

Question 4. For each of the graphs in Figure 1

(a) Find the degree of each vertex.

(b) Use the results from 1 to determine if the graph is Eulerian, semi-Eulerian, or neither.

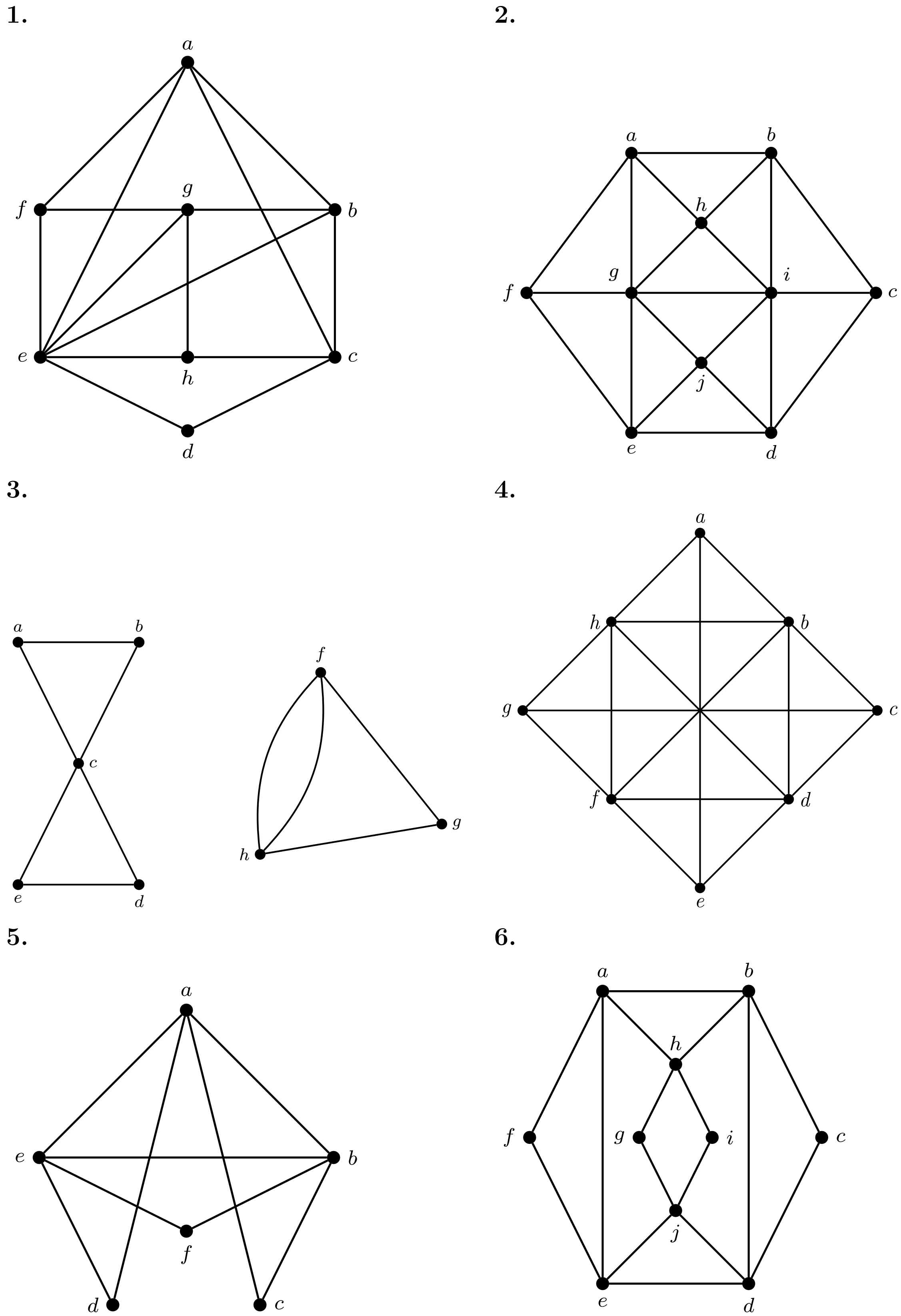


Figure 1

Solution.

1. The graph is semi-Eulerian

$$\deg(a) = 4, \quad \deg(b) = 4, \quad \deg(c) = 4, \quad \deg(d) = 2, \quad \deg(e) = 6,$$

$$\deg(f) = 3, \quad \deg(g) = 4, \quad \deg(h) = 3$$

2. The graph is semi-Eulerian

$$\deg(a) = 4, \quad \deg(b) = 4, \quad \deg(c) = 3, \quad \deg(d) = 4,$$

$$\deg(e) = 4, \quad \deg(f) = 3, \quad \deg(g) = 6, \quad \deg(h) = 4, \quad \deg(i) = 6, \quad \deg(j) = 4$$

3. The graph is neither Eulerian nor semi-Eulerian

$$\deg(a) = 2, \quad \deg(b) = 2, \quad \deg(c) = 4, \quad \deg(d) = 2, \quad \deg(e) = 2,$$

$$\deg(f) = 3, \quad \deg(g) = 2, \quad \deg(h) = 3$$

4. The graph is neither Eulerian nor semi-Eulerian

$$\deg(a) = 3, \quad \deg(b) = 5, \quad \deg(c) = 3, \quad \deg(d) = 5, \quad \deg(e) = 3,$$

$$\deg(f) = 5, \quad \deg(g) = 3, \quad \deg(h) = 5$$

5. The graph is Eulerian

$$\deg(a) = 4, \quad \deg(b) = 4, \quad \deg(c) = 2, \quad \deg(d) = 2, \quad \deg(e) = 4, \quad \deg(f) = 2$$

6. The graph is Eulerian

$$\deg(a) = 4, \quad \deg(b) = 4, \quad \deg(c) = 2, \quad \deg(d) = 4, \quad \deg(e) = 4, \quad \deg(f) = 2,$$

$$\deg(g) = 2, \quad \deg(h) = 4, \quad \deg(i) = 2, \quad \deg(j) = 4$$

Question 5. For those graphs in Figure 1 that have one, find an Eulerian circuit or Eulerian trail.

Solution.

1. An Eulerian trail is

$$f,\ a,\ b,\ c,\ a,\ e,\ b,\ g,\ e,\ d,\ c,\ h,\ e,\ f,\ g,\ h$$

2. An Eulerian trail is

$$f,\ a,\ b,\ c,\ d,\ e,\ f,\ g,\ a,\ h,\ b,\ i,\ d,\ j,\ e,\ g,\ h,\ i,\ g,\ j,\ i,\ c$$

3. The graph is neither Eulerian nor semi-Eulerian.

4. The graph is neither Eulerian nor semi-Eulerian

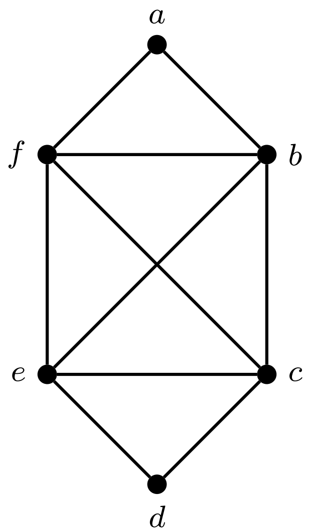
5. An Eulerian circuit is
$$a,\ b,\ c,\ a,\ d,\ e,\ b,\ f,\ e,\ a$$

6. An Eulerian circuit is

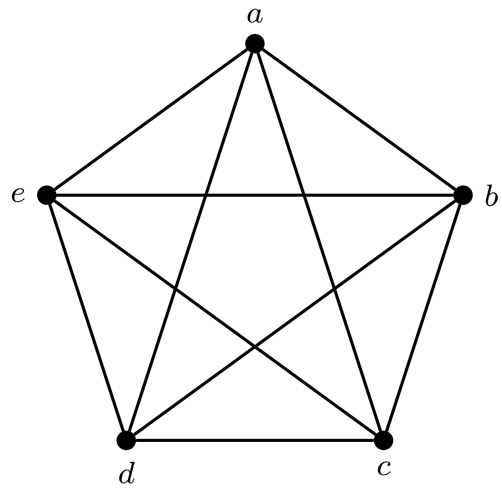
$$a,\ b,\ c,\ d,\ b,\ h,\ a,\ e,\ d,\ j,\ g,\ h,\ i,\ j,\ e,\ f,\ a$$

Question 6. Find an Eulerian circuit or Eulerian trail for each of the graphs below

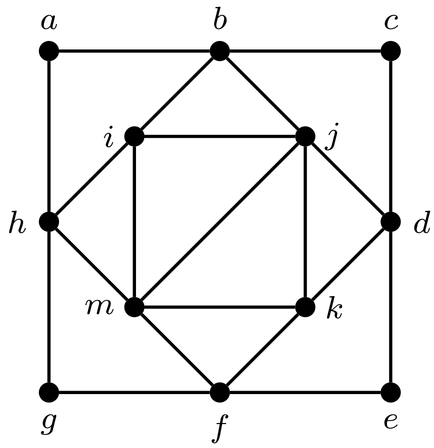
1.



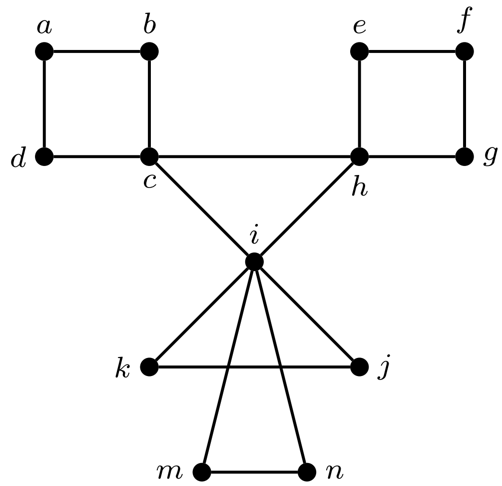
2.



3.



4.



Solution.

1. An Eulerian circuit is

$$a,\ b,\ c,\ d,\ e,\ b,\ f,\ c,\ e,\ f,\ a$$

2. An Eulerian circuit is

$$a,\ b,\ c,\ a,\ d,\ b,\ e,\ c,\ d,\ e,\ a$$

3. An Eulerian trail is

$$j,\ b,\ a,\ h,\ g,\ f,\ e,\ d,\ c,\ b,\ i,\ h,\ m,\ f,\ k,\ d,\ j,\ i,\ m,\ j,\ k,\ m$$

4. An Eulerian circuit is

$$a,\ b,\ c,\ h,\ g,\ f,\ e,\ h,\ i,\ k,\ j,\ i,\ m,\ n,\ i,\ c,\ d,\ a$$

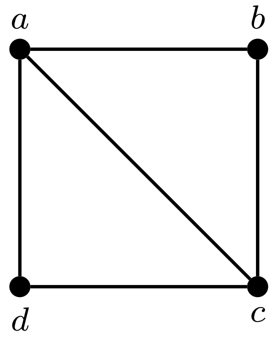
Question 7. For each of the following graphs G defined by its edge set, construct its graphical representation. Determine whether G admits an Eulerian circuit or an Eulerian path. If yes, find it using Fleury's algorithm or Hierholzer's algorithm.

1. $E(G) = \{ab, bc, cd, da, ac\}$
2. $E(G) = \{ab, bc, cd, da, bd\}$
3. $E(G) = \{ab, bc, cd, da, ac, bd, be\}$
4. $E(G) = \{mn, no, op, pm, mo, np, pq\}$
5. $E(G) = \{pq, qr, rs, st, tp, pr, qs\}$
6. $E(G) = \{ab, bc, cd, de, ea, bd, ce\}$
7. $E(G) = \{ab, bc, cd, de, ef, fg, gh, hi, ij, jk, kl, la, ag, bh, ci, dj, ek, fl\}$
8. $E(G) = \{ab, bc, cd, de, ef, fa, ag, bh, ci, dj, ek, fl, gh, hi, ij, jk, kl, la, aj, bi, ch, dg\}$
9. $E(G) = \{ab, bc, cd, de, ef, fg, gh, ha, ac, bd, ce, df, eg, fh, ga, hb, ad, be, cf, dg, eh, fa, gb, hc\}$
10. $E(G) = \{ab, bc, cd, de, ef, fa, gh, hi, ij, jg, kl, lm, mn, nk, ag, gk, bh, hl, ci, im, dj\}$

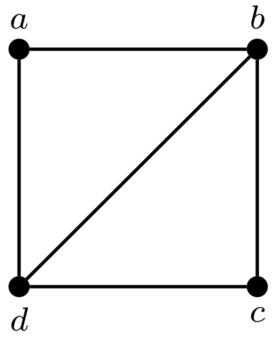
Solution.

1. An Eulerian trail: ab, bc, ca, ad, dc
2. An Eulerian trail: bc, cd, da, ab, bd
3. There are four odd vertices, the graph is neither Eulerian nor semi-Eulerian
4. There are four odd vertices, the graph is neither Eulerian nor semi-Eulerian
5. There are four odd vertices, the graph is neither Eulerian nor semi-Eulerian
6. There are four odd vertices, the graph is neither Eulerian nor semi-Eulerian
7. The graph is neither Eulerian nor semi-Eulerian
8. The graph is neither Eulerian nor semi-Eulerian
9. An Eulerian circuit:
$$a,\ b,\ c,\ d,\ e,\ f,\ g,\ h,\ a,\ c,\ e,\ g,\ a,\ d,\ b,\ h,\ f,\ d,\ g,\ b,\ e,\ h,\ c,\ f,\ a$$
10. The graph is neither Eulerian nor semi-Eulerian

1.



2.



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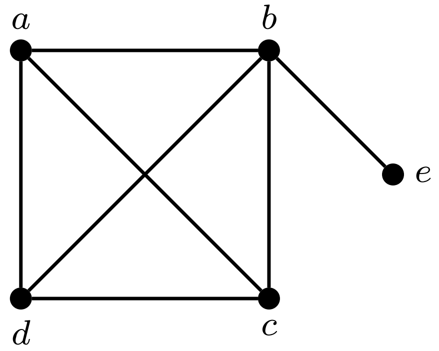

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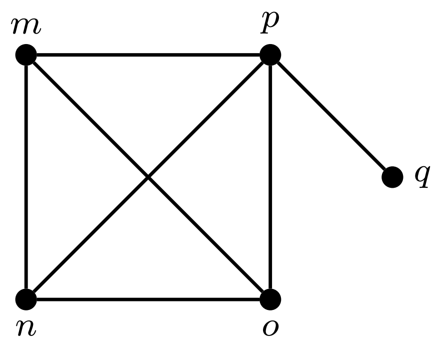

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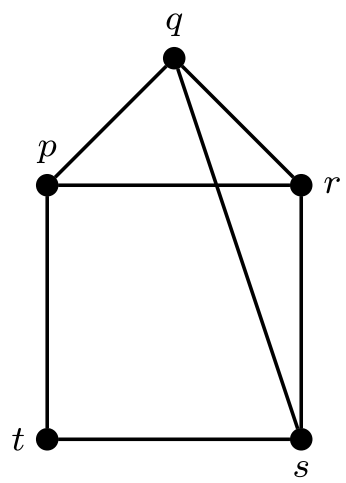

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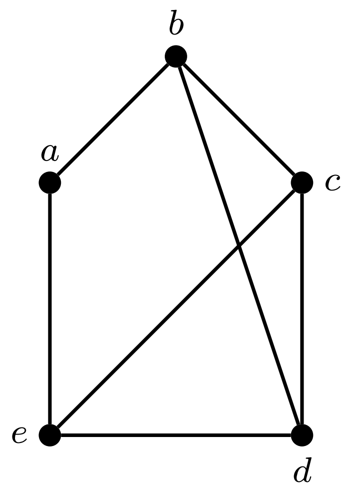

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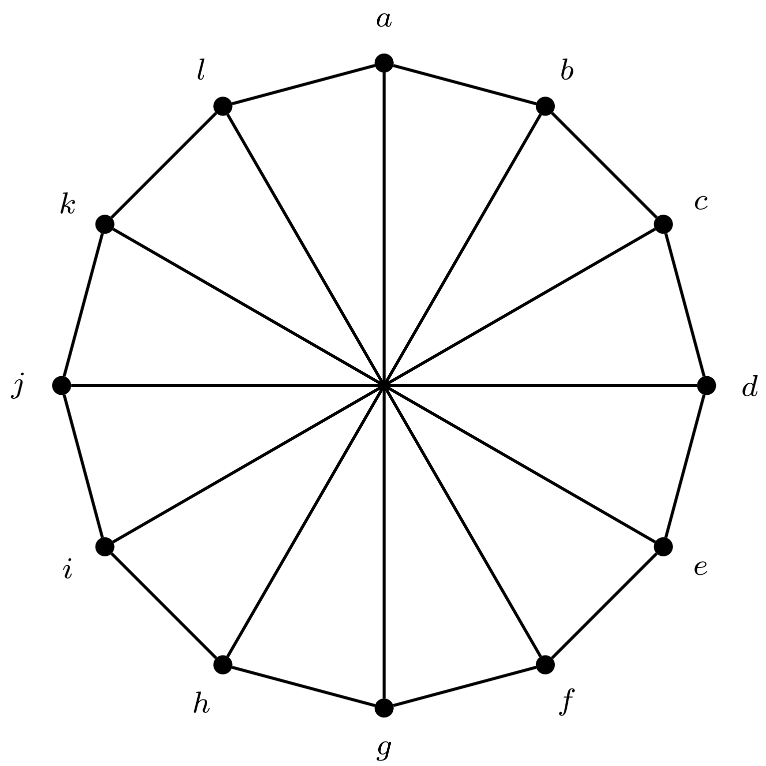

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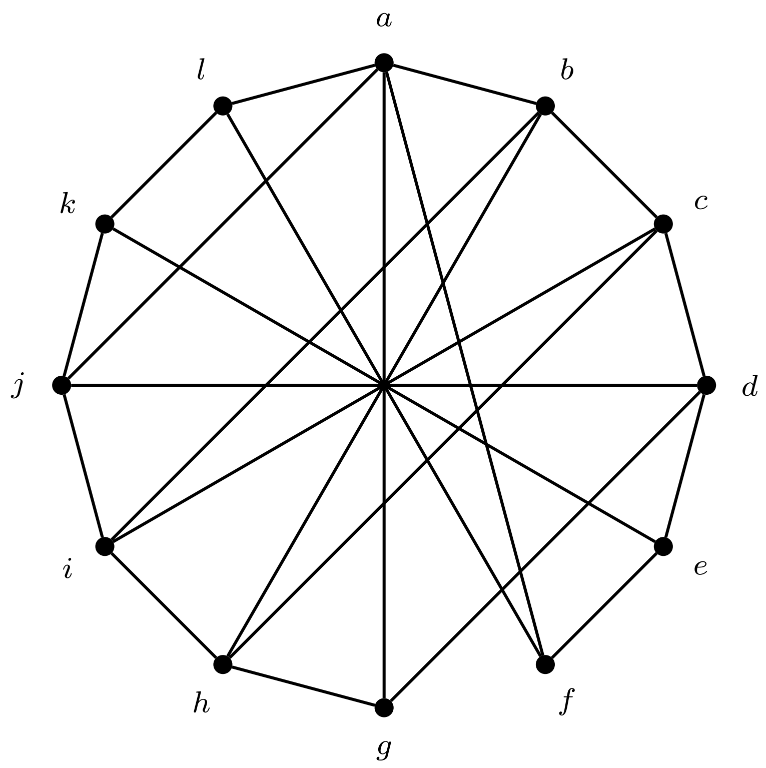

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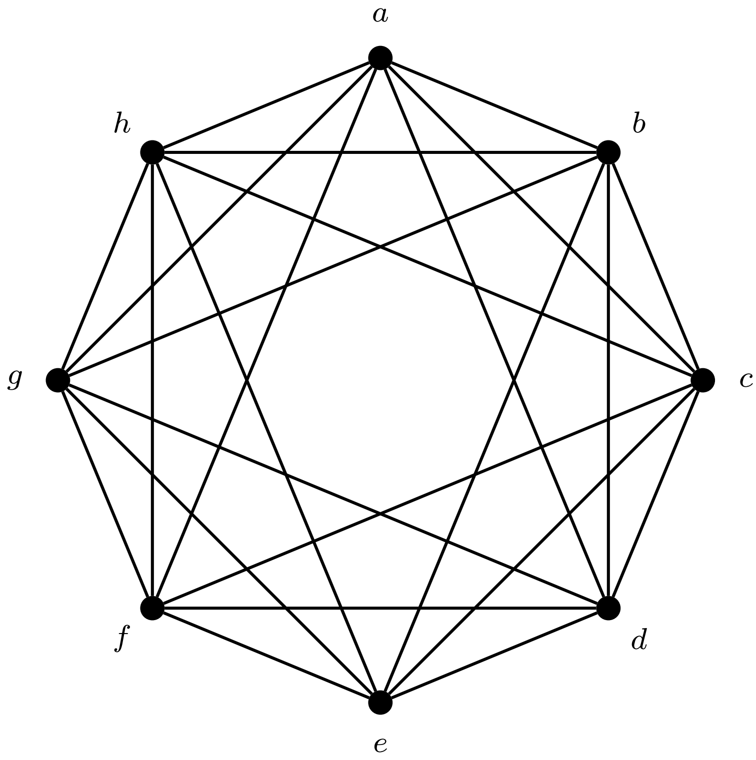
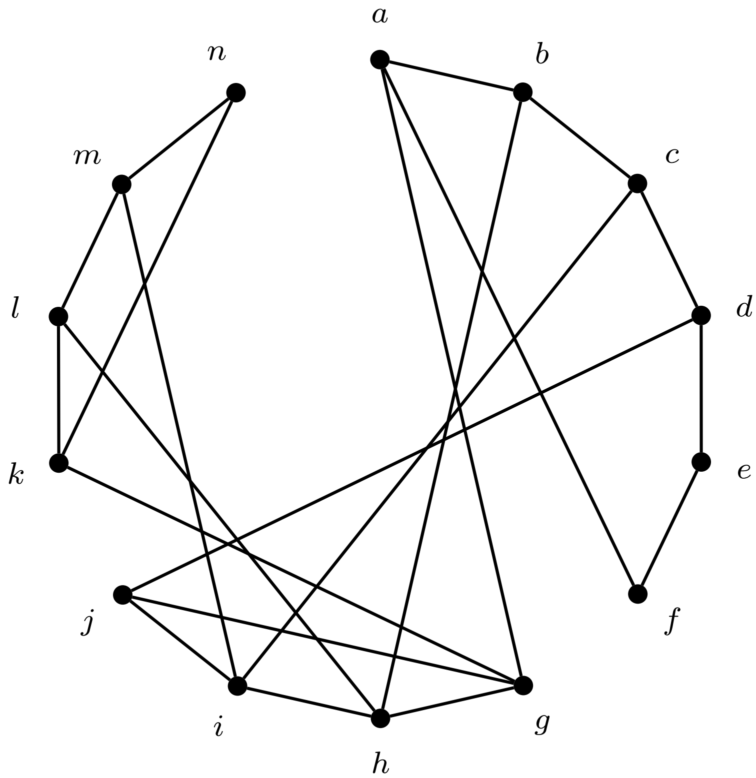


10.



Question 8. For each of the following graph types, determine whether an Eulerian circuit or path is possible and justify your conclusions.

1. A complete graph.
2. A tree - a connected graph that contains no cycles or circuits

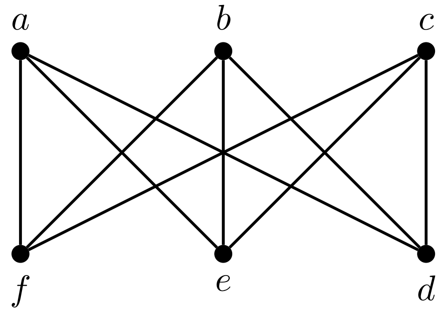
3. A graph containing at least one bridge, where a *bridge* is an edge whose removal increases the number of connected components in the graph.

Solution.

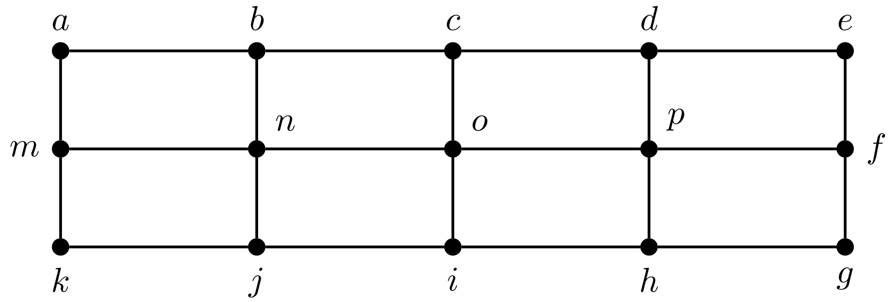
1. In a complete graph K_n with n vertices, each vertex has degree $n-1$.
   - If $n = 2$, the graph has two vertices with one edge and has an Eulerian trail.
   - If $n > 2$ is even, then each vertex has an odd degree, meaning an Eulerian circuit and an Eulerian trail do not exist.
   - If n is odd, then each vertex has an even degree, meaning an Eulerian circuit exists.
2. Since a tree has no cycles, it cannot contain an Eulerian circuit. An Eulerian trail might exist.
3. If a graph contains a bridge, then any Eulerian circuit must traverse the bridge exactly twice—once in each direction. This contradicts the requirement that each edge must be traversed exactly once, so an Eulerian circuit does not exist. An Eulerian trail might exist.

Question 9. Find an optimal Eulerization and semi-Eulerization for each of the graphs below

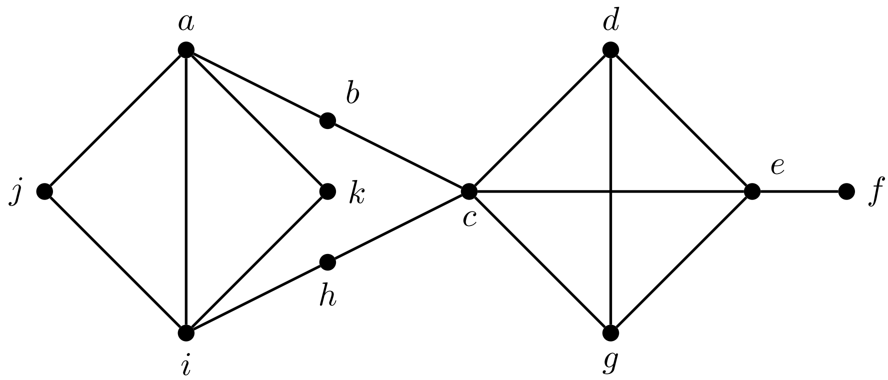
1.



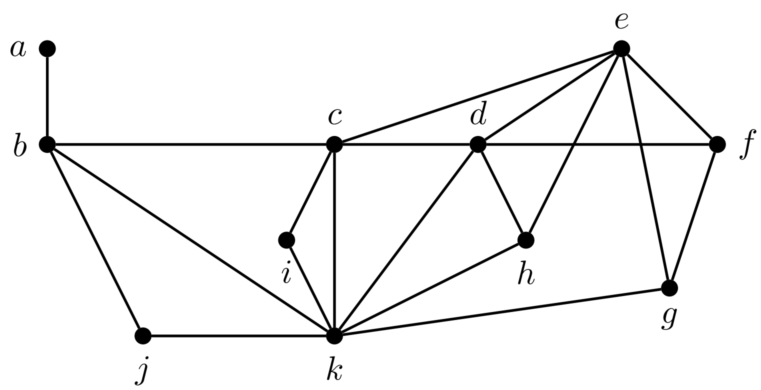
2.



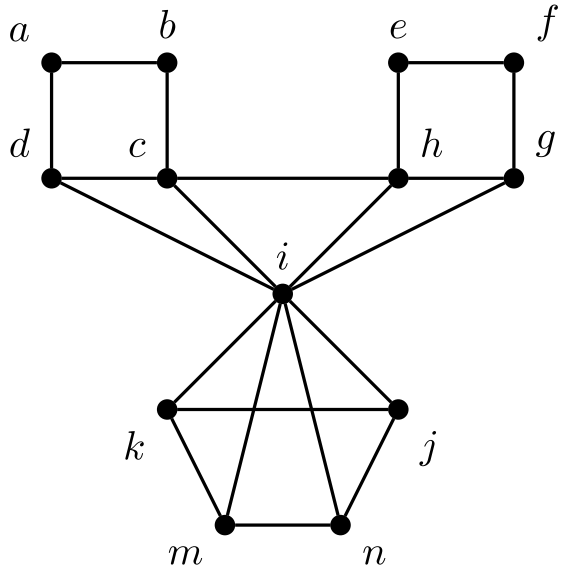
3.



4.



5.



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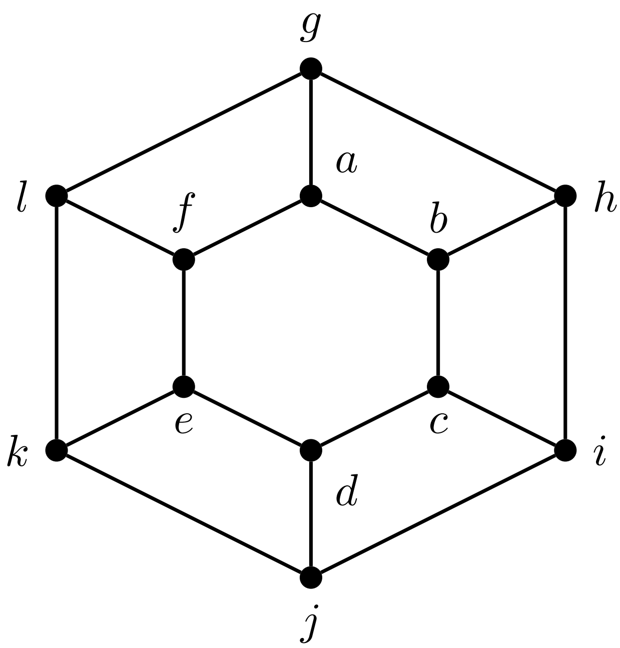

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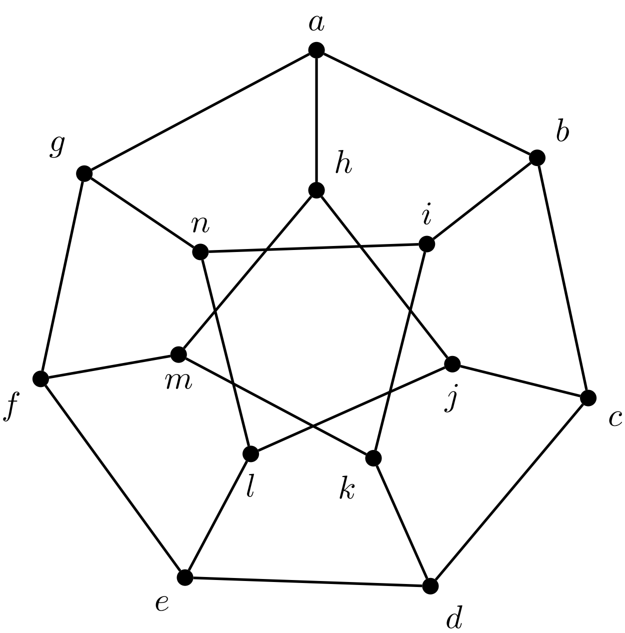

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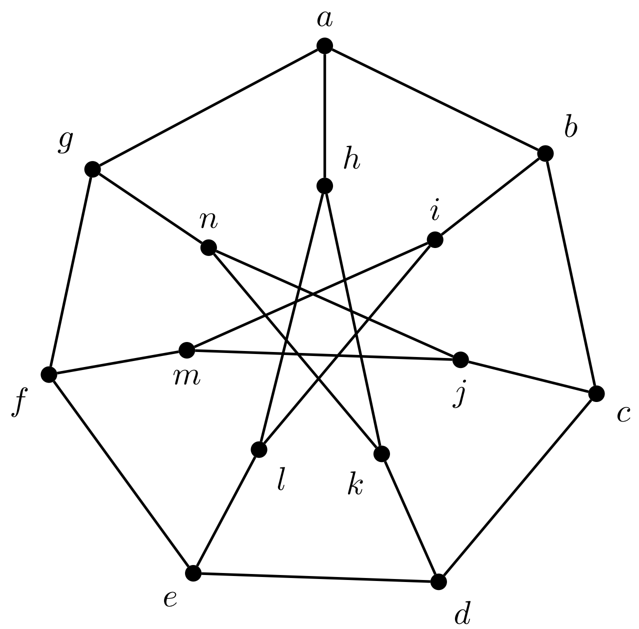
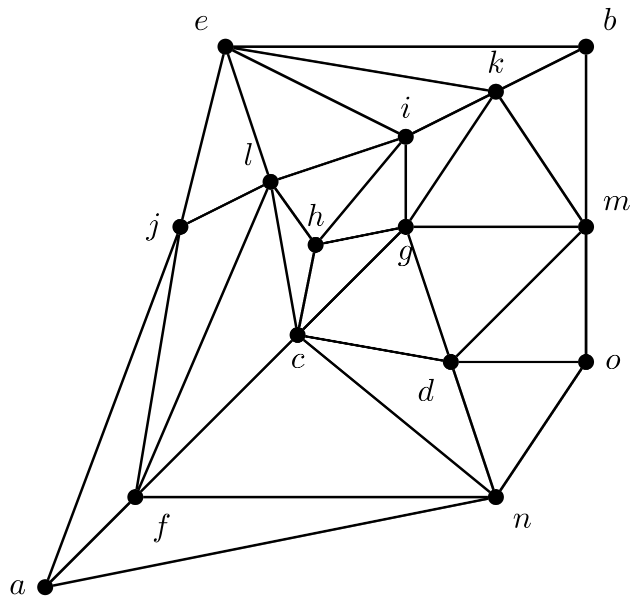
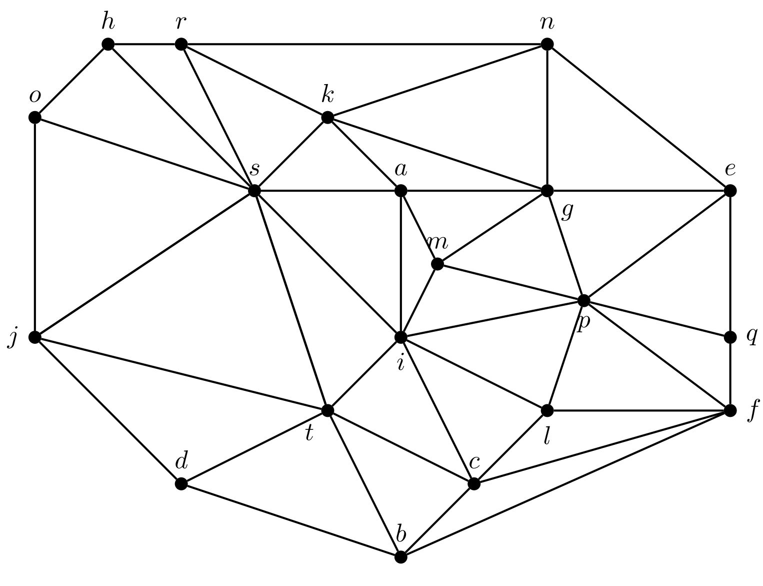


9.



10.



Solution.

1. In this graph, all six vertices are odd. To Eulerize the graph, we can pair them into three adjacent pairs, allowing us to add three edges—for example, ad, eb, and fc—to create an Eulerian circuit.

   For semi-Eulerization, we duplicate two edges to ensure that only two vertices remain with an odd degree. A possible choice is to duplicate af and be.

2. Eulerization: ab, am, cd, pf, ph, ji.
   Semi-Eulerization: cd, ih, jb.

3. Eulerization: ce, dg, ef.
   Semi-Eulerization: cd.

4. Eulerization: ab, bc, kg, df, eh.
   Semi-Eulerization: ck, de, fg.

5. Eulerization: di, gi, km, jn.
   Semi-Eulerization: mk, jn.

6. Eulerization: ag, bh, ci, dj, ek, fl.
   Semi-Eulerization: bc, af, gh, lk, dj.

7. Eulerization: $ah, bi, cj, dk, el, fm, gn$.
   Semi-Eulerization: cd, ef, gn, bi, km, ah.

8. Eulerization: $ah, bi, cj, dk, el, fm, gn$.
   Semi-Eulerization: im, cd, ef, gn, hk, ba.

9. Eulerization: li, cn, cg, om.
   Semi-Eulerization: dc, ek, fn, mo.

10. Eulerization: ak, cf, ti, dj, pq, ho.
    Semi-Eulerization: qf, ak, ci, oh.

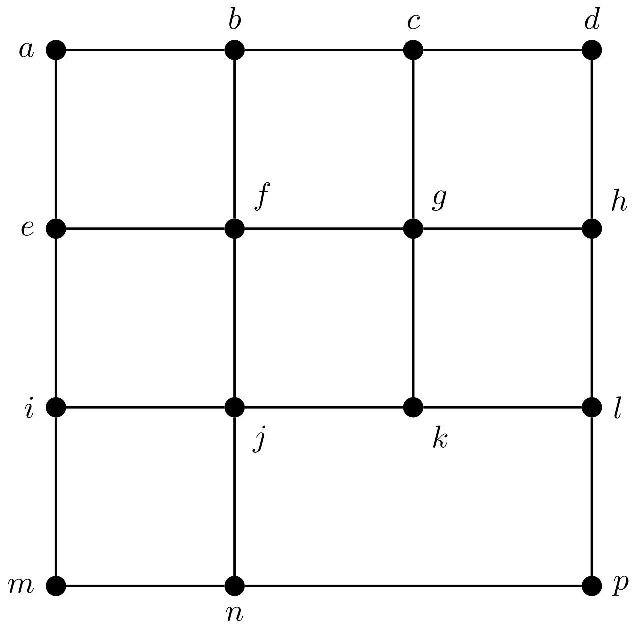
Use the following map for questions 10 and 11

Question 10. Sarah is planning the route for the neighborhood watch night patrol for the community shown in the map above. The person on patrol must walk along each street at least once, including the perimeter of the park.

1. Model this scenario as a graph
2. Determine if the graph is Eulerian or semi-Eulerian or neither. Eulerize the graph if it is not Eulerian
3. Find an Eulerian circuit starting and ending at the post office

Solution.

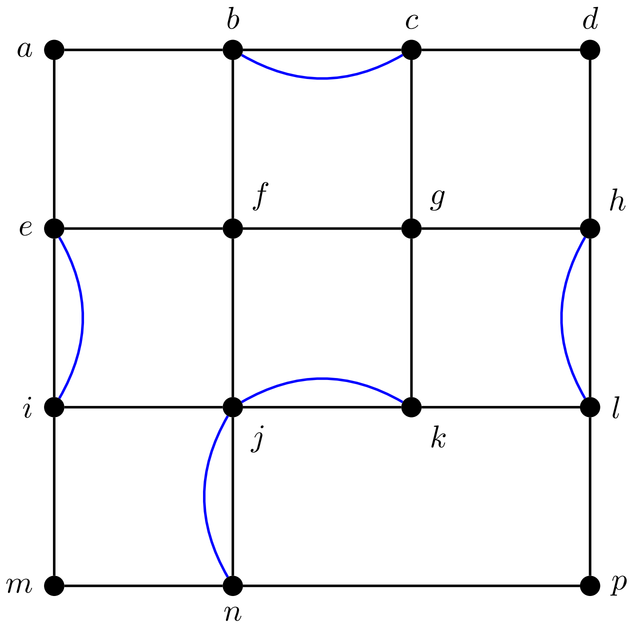
1. A graph model of this scenario is as follows, where vertices represent intersections and edges represent street blocks



2. The graph is neither Eulerian nor semi-Eulerian. An Eulerization of the graph can be achieved by duplicating the following edges:

$$bc,\ ei,\ hl,\ jk,\ jn$$

The Eulerized graph is shown below



3. The vertex j corresponds to the post office. An Eulerian circuit starting and ending at the vertex j is given by

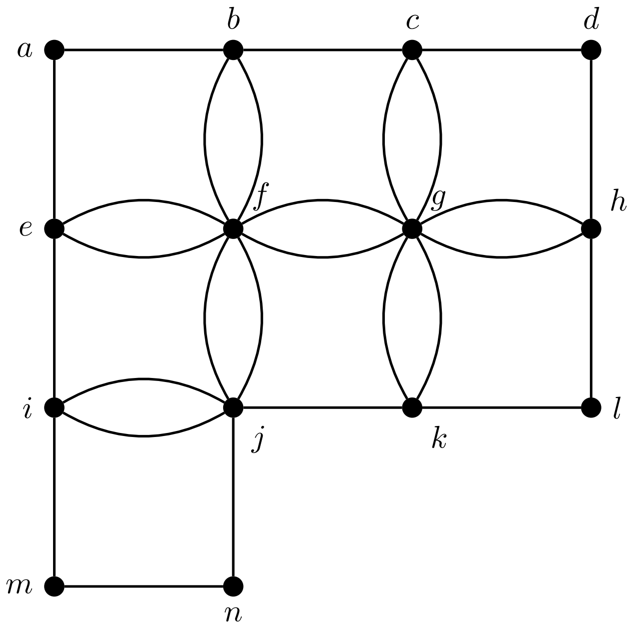
$$j,\ k,\ l,\ h,\ l,\ p,\ o,\ n,\ j,\ f,\ b,\ c,\ g,\ k,\ o,\ n,\ m,\ i,\ e,\ a,\ b,\ c,\ d,\ h,\ g,\ f,\ e,\ i,\ j$$

Question 11. A postal worker is delivering mail along her route shown in the map above. She must walk down both sides of the street if there are houses on both sides and does not need to walk the streets that only border the park (since no houses are in the park).

1. Model this scenario as a graph (Hint: make use of multi-edges).
2. Determine if the graph is Eulerian or semi-Eulerian or neither. Eulerize the graph if it is not Eulerian.
3. Find an Eulerian circuit starting and ending at the post office

Solution.

1. A graph model of this scenario is as follows, where vertices represent intersections and edges represent street blocks

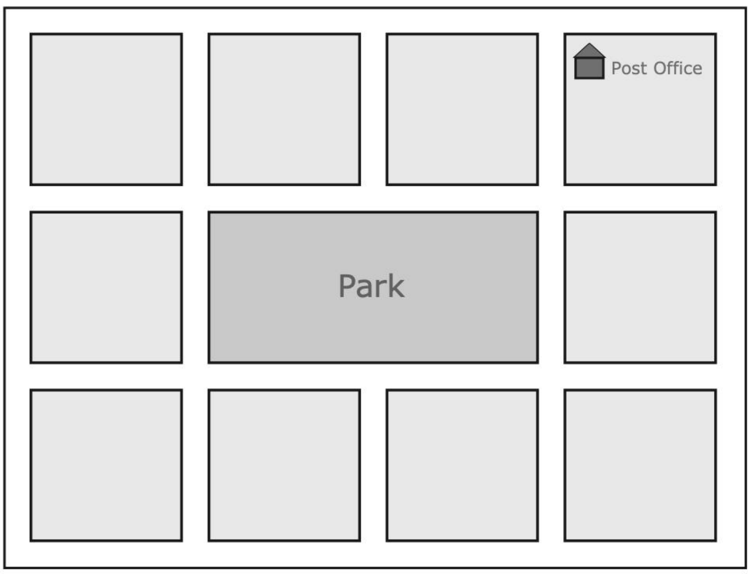


2. The graph is Eulerian.

3. The vertex j corresponds to the post office. An Eulerian circuit starting and ending at the vertex j is given by

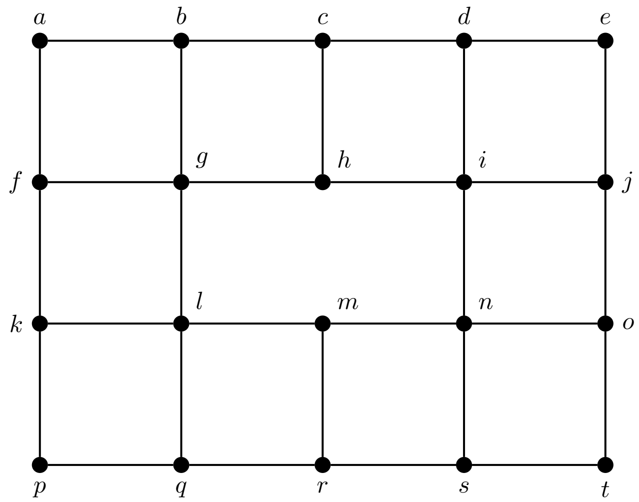
$$j,\ i,\ j,\ f,\ b,\ c,\ g,\ k,\ g,\ c,\ d,\ h,\ l,\ k,\ j,\ n,\ m,\ i,\ e,\ f,\ g,\ h,\ g,\ f,\ e,\ a,\ b,\ f,\ j$$

Question 12. Repeat Question 10 with the map below.



Solution.

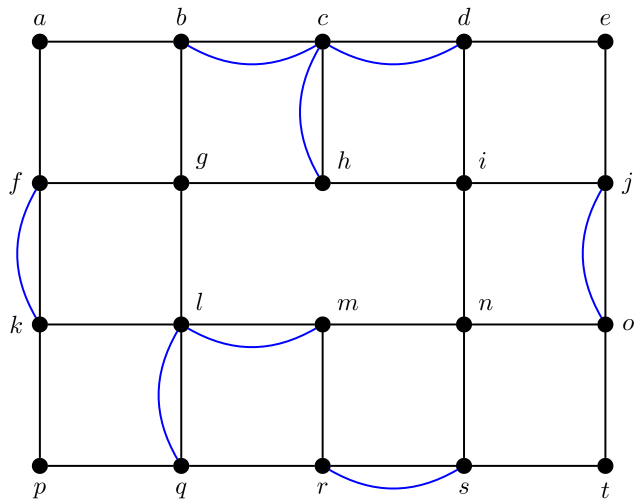
1. A graph model of this scenario is as follows, where vertices represent intersections and edges represent street blocks



2. The graph is neither Eulerian nor semi-Eulerian. An Eulerization of the graph can be achieved by duplicating the following edges:

$$bc,\ cd,\ ch,\ fk,\ jo,\ lm,\ lq,\ rs$$

The Eulerized graph is shown below



3. The vertex d corresponds to the post office. An Eulerian circuit starting and ending at the vertex d is given by

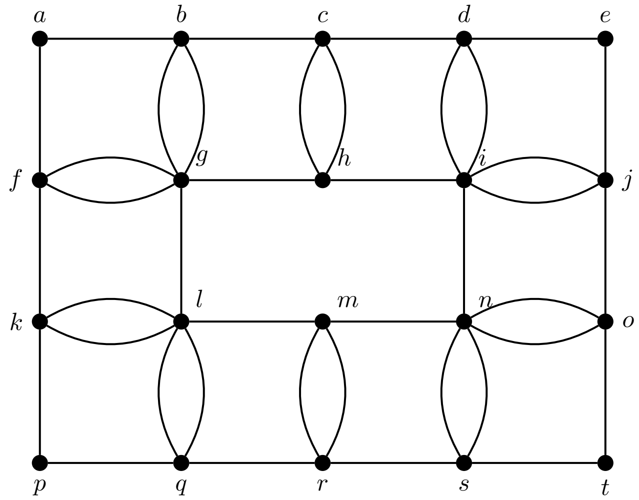
$$d,\ e,\ j,\ i,\ h,\ g,\ f,\ k,\ l,\ m,\ n,\ o,\ j,\ o,\ t,\ s,\ r,\ m,\ h,\ c,\ d,\ i,\ n,\ s,\ r,\ q,\ l,\ g,\ b,\ g,\ l,\ q,$$

$$p,\ k,\ f,\ a,\ b,\ c,\ d$$

Question 13. Repeat Question 11 with the map above.

Solution.

1. A graph model of this scenario is as follows, where vertices represent intersections and edges represent street blocks



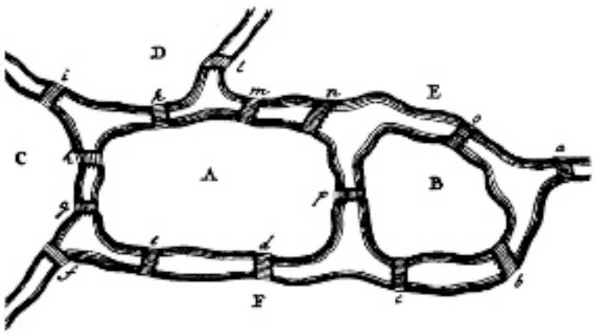
2. The graph is Eulerian.

3. The vertex d corresponds to the post office. An Eulerian circuit starting and ending at the vertex d is given by

$$d,\ e,\ j,\ i,\ h,\ c,\ h,\ g,\ b,\ g,\ f,\ g,\ l,\ m,\ n,\ o,\ n,\ s,\ t,\ o,\ j,\ i,\ d,\ i,\ n,\ s,\ r,\ m,\ r,$$

$$q,\ p,\ k,\ l,\ q,\ l,\ k,\ f,\ a,\ b,\ c,\ d$$

Question 14. The image below appeared in Euler's original paper on Königsberg bridge problem[1]



1. Model the map as a graph (Hint: make use of multi-edges).

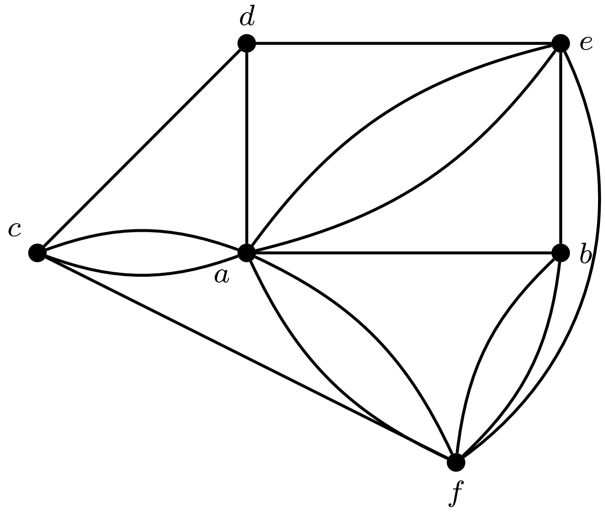
2. Determine if the graph is Eulerian or semi-Eulerian or neither. Eulerize the graph if it is not Eulerian.

3. Find an Eulerian circuit for either the original graph or its Eulerization.

Solution.

1. A graph model of the map is as follows, where vertices represent landmasses and edges represent bridges

[1]Euler, Leonhard. *Solutio problematis ad geometriam situs pertinentis.* Commentarii academiae scientiarum Petropolitanae (1741): 128-140.



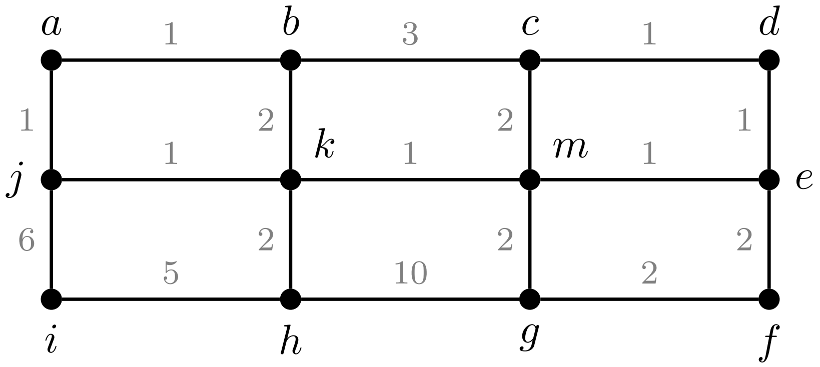
2. The graph is neither Eulerian nor semi-Eulerian. An Eulerization of the graph can be achieved by duplicating the edge ed

3. An Eulerian circuit for the Eulerized graph is given by

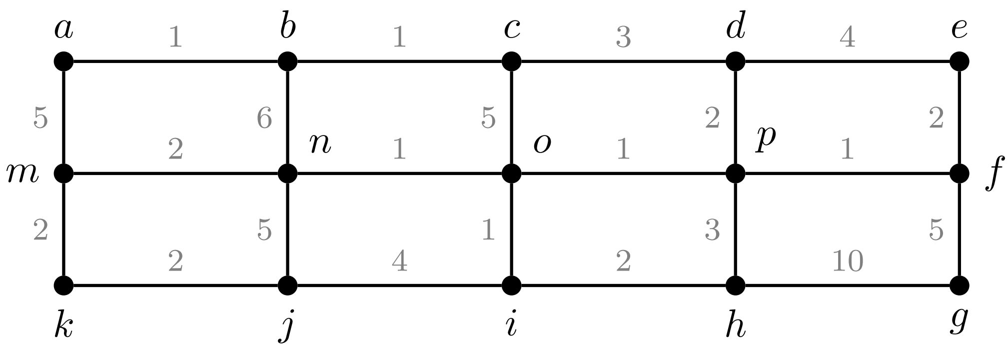
$$e,\ d,\ c,\ f,\ e,\ d,\ a,\ b,\ e,\ a,\ c,\ a,\ f,\ b,\ f,\ a,\ e$$

Question 15. Find an optimal Eulerization and semi-Eulerization for each of the weighted graphs below.

1.

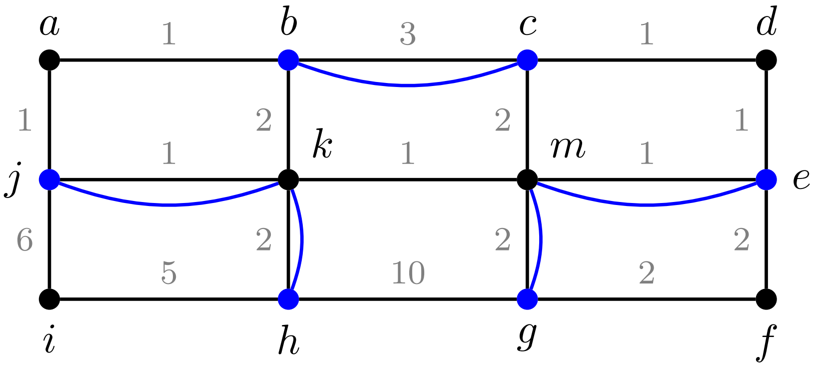


2.

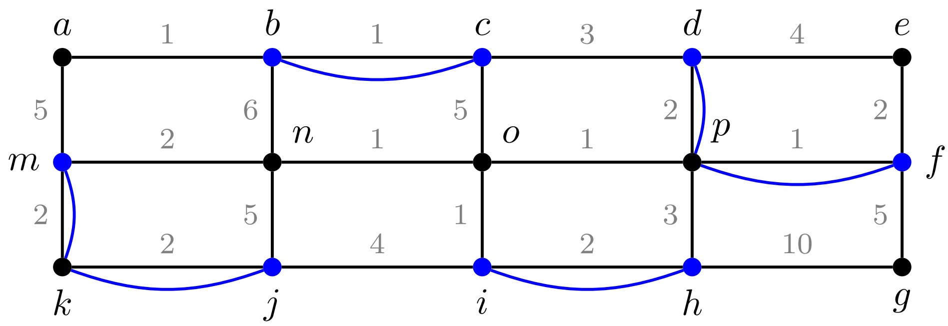


Solution.

1.



2.



Question 16. Fleury's Algorithm can be used to find an Eulerian circuit or an Eulerian trail, yet Hierholzer's is written in such a way as to only work for Eulerian circuits. Modify Hierholzer's Algorithm so it can be used to find an Eulerian trail. (Hint: think about how Fleury's accounts for the two odd vertices.)

Solution.

1. *Identify the odd-degree vertices:* Determine the two vertices, say u and v, that have an odd degree. These will serve as the starting and ending points of the Eulerian trail.

2. *Introduce a temporary edge:* Add an artificial edge between u and v, effectively making all vertices have even degree. This ensures that the modified graph is Eulerian.

3. *Apply Hierholzer's Algorithm:* Run the standard Hierholzer's Algorithm on the modified graph, starting at u. When constructing the first cycle, first traverse the artificially added edge uv.

4. *Extract the Eulerian trail:* The resulting Eulerian circuit in the modified graph will start and end at u, with exactly one traversal of uv. To recover the Eulerian trail in the original graph, remove the artificial edge uv, then traverse the remaining edges in reverse order in the circuit. This yields an Eulerian trail that starts at u and ends at v.

Question 17. A connected graph G has 8 vertices and 15 edges. You know three vertices have degree 5 and three vertices have degree 3. What are the possible degrees for the remaining two vertices?

Solution. Since the number of edges is 15, the total degree must be 30. The 6 vertices give a total degree of 24 and so the last two vertices have a degree total of 6. Since there must be an odd number of odd vertices, either both or neither of the remaining vertices must have an odd degree. The possible degrees are 1 and 5, 2 and 4, and 3 and 3. It cannot be 0 and 6 since the graph is connected.

Enrichment Questions

Question 18. Implement an algorithm in a programming language of your choice (e.g., Python, Mathematica, or C++) to determine whether a given graph contains an Eulerian trail or an Eulerian circuit. If such a trail or circuit exists, the algorithm should output the corresponding sequence of edges; otherwise, it should indicate that no Eulerian trail/circuit exists.

1. Use Fleury's algorithm or Hierholzer's algorithm.
2. Test the implementation on different example graphs and determine whether each graph has an Eulerian path or circuit.

Question 19. *Application of Eulerian circuits: a delivery service case study* Consider a scenario in which a delivery service seeks to optimize its routes by minimizing travel distance while ensuring that each street is traversed exactly once. The city streets are represented as a graph, where intersections correspond to vertices and streets between them correspond to edges. Your task is to:

1. Determine whether a route exists that satisfies this requirement.
2. If such a route exists, construct it. If no such route is possible, propose an alternative strategy that allows the delivery driver to traverse all streets while minimizing the number of repeated edges.

You may approach this problem in one of the following ways:

- Formulate a general solution applicable to any city layout.
- Apply the solution to your hometown (or a region with at least 20 vertices).
- Solve the problem specifically for Bratislava, ensuring that at least 20 vertices are considered.

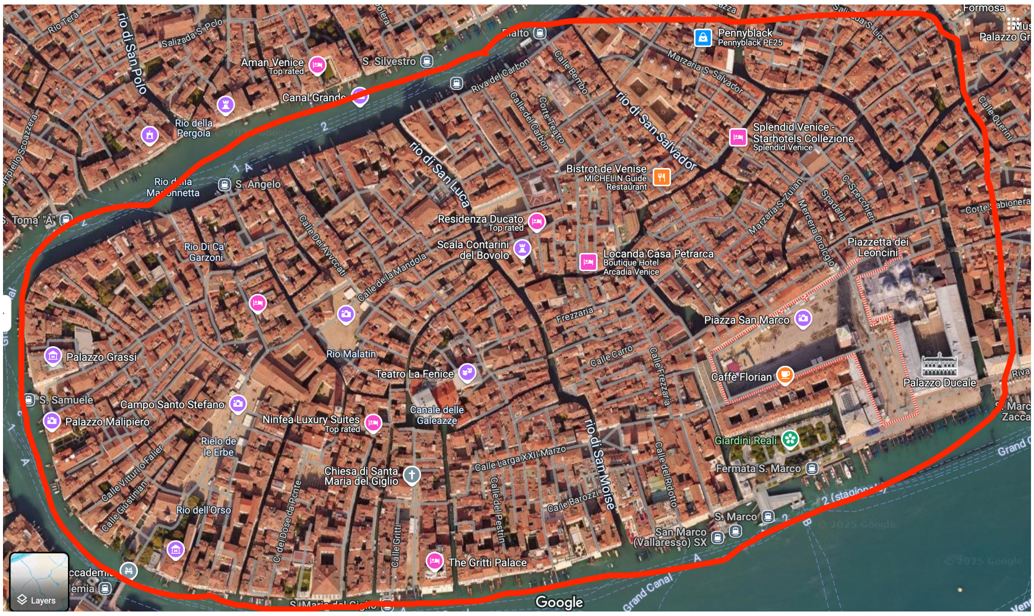
For practical implementation, use Google Maps as a reference.

Question 20. Identify and describe three real-world scenarios, distinct from those mentioned below, that can be modeled using Eulerian paths or circuits. For each case, describe why this model is appropriate, and propose a viable solution. Additionally, provide a simplified implementation of the model with at least 10 vertices.

Potential Applications:

1. Optimization of waste collection routes in urban areas.
2. Scheduling and maintenance of railway infrastructure.
3. Analysis and optimization of electrical energy distribution in power grids.

Question 21. The image represents a map of part of Venice, with the area of interest marked by a red outline. A standard Google Map is used; if you require more details, you can refer to the original source.



1. Model the map as a graph, where vertices represent landmasses (islands and riverbanks) and edges represent bridges connecting them. Include all bridges in your graph.
2. Determine if the graph is Eulerian or semi-Eulerian or neither. Eulerize the graph if it is not Eulerian
3. Find an Eulerian circuit either in the original graph or in its Eulerized version.

If you prefer not to solve the task manually, implement a program to do so.

Question 22. You have been hired to create a route for the bridge inspector for the city of Pittsburgh, Pennsylvania, which is known for its bridges. The city needs the bridges to be visually inspected on the first day of every month and so needs the route to be as short as possible in order for the inspector to complete his tour in one day.

1. Use a high quality map to create a graph similar to the one that arose from Königsberg.
2. Add edge weights that correspond to distance or time and find an optimal exhaustive route (a tour through the graph that visits each bridge at least once).
3. Write a detailed report for the Pittsburgh city manager outlining your methodology, results, and recommendations for the bridge inspector.

Question 23. Pick a city neighborhood. Using a quality map, model the neighborhood as a graph and assign weights to the edges. You may use any logical metric in assigning weights (such as time or distance). Find an optimal route for a street sweeper that must visit each street in the neighborhood.