Tutorial 7

Fundamental spaces and decompositions

Question 1. Determine whether $\boldsymbol{b}$ is in the column space of A, and if so, express $\boldsymbol{b}$ as a linear combination of the column vectors of A

1. $A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{pmatrix}, \quad \boldsymbol{b} = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$

2. $A = \begin{pmatrix} 1 & -1 & 1 \\ 9 & 3 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad \boldsymbol{b} = \begin{pmatrix} 5 \\ 1 \\ -1 \end{pmatrix}$

3. $A = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}, \quad \boldsymbol{b} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$

4. $A = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 2 & 1 \\ 1 & 2 & 1 & 3 \\ 0 & 1 & 2 & 2 \end{pmatrix}, \quad \boldsymbol{b} = \begin{pmatrix} 4 \\ 3 \\ 5 \\ 7 \end{pmatrix}$

Solution. $\boldsymbol{b}$ is in the column space of A iff $A\boldsymbol{x} = \boldsymbol{b}$ for some vector $\boldsymbol{x}$. Therefore, for each case, we solve for $\boldsymbol{x}$ in the linear system $A\boldsymbol{x} = \boldsymbol{b}$.

1. The reduced row echelon form of the augmented matrix is:

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

The system does not have a solution and $\boldsymbol{b}$ does not belong to the column space of A.

2. The reduced row echelon form of the augmented matrix is:

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \end{array}\right).$$

The system has a unique solution $\begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix}$. We have

$$\boldsymbol{b} = \begin{pmatrix} 1 \\ 9 \\ 1 \end{pmatrix} - 3\begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

3. The reduced row echelon form of the augmented matrix is:

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

The system does not have a solution and $\boldsymbol{b}$ does not belong to the column space of A.

4. The reduced row echelon form of the augmented matrix is:

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -26 \\ 0 & 1 & 0 & 0 & 13 \\ 0 & 0 & 1 & 0 & -7 \\ 0 & 0 & 0 & 1 & 4 \end{array}\right).$$

The system has a unique solution $\begin{pmatrix} -26 \\ 13 \\ -7 \\ 4 \end{pmatrix}$. We have

$$\boldsymbol{b} = -26\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + 13\begin{pmatrix} 2 \\ 1 \\ 2 \\ 1 \end{pmatrix} - 7\begin{pmatrix} 0 \\ 2 \\ 1 \\ 2 \end{pmatrix} + 4\begin{pmatrix} 1 \\ 1 \\ 3 \\ 2 \end{pmatrix}$$

Question 2. Suppose that $x_1 = 3, x_2 = 0, x_3 = -1, x_4 = 5$ is a solution of a nonhomogeneous linear system $A\boldsymbol{x} = \boldsymbol{b}$ and that the solution set of the homogeneous system $A\boldsymbol{x} = \boldsymbol{0}$ is given by the formulas

$$x_1 = 5r - 2s, \quad x_2 = s, \quad x_3 = s + t, \quad x_4 = t$$

1. Find a vector form of the general solution of $A\boldsymbol{x} = \boldsymbol{0}$.
2. Find a vector form of the general solution of $A\boldsymbol{x} = \boldsymbol{b}$.

Solution.

1. A vector form of the general solution of $A\boldsymbol{x} = \boldsymbol{0}$ is given by

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = r\begin{pmatrix} 5 \\ 0 \\ 0 \\ 0 \end{pmatrix} + s\begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

2. A vector form of the general solution of $A\boldsymbol{x} = \boldsymbol{b}$ is given by

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ -1 \\ 5 \end{pmatrix} + r\begin{pmatrix} 5 \\ 0 \\ 0 \\ 0 \end{pmatrix} + s\begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

Question 3. Suppose that $x_1 = -1, x_2 = 2, x_3 = 4, x_4 = -3$ is a solution of a nonhomogeneous linear system $A\boldsymbol{x} = \boldsymbol{b}$ and that the solution set of the homogeneous system $A\boldsymbol{x} = \boldsymbol{0}$ is given by the formulas

$$x_1 = -3r + 4s, \quad x_2 = r - s, \quad x_3 = r, \quad x_4 = s$$

1. Find a vector form of the general solution of $A\boldsymbol{x} = \boldsymbol{0}$.
2. Find a vector form of the general solution of $A\boldsymbol{x} = \boldsymbol{b}$.

Solution.

1. A vector form of the general solution of $A\boldsymbol{x} = \boldsymbol{0}$ is given by

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = r\begin{pmatrix} -3 \\ 1 \\ 1 \\ 0 \end{pmatrix} + s\begin{pmatrix} 4 \\ -1 \\ 0 \\ 1 \end{pmatrix}$$

2. A vector form of the general solution of $A\boldsymbol{x} = \boldsymbol{b}$ is given by

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 4 \\ -3 \end{pmatrix} + r \begin{pmatrix} -3 \\ 1 \\ 1 \\ 0 \end{pmatrix} + s \begin{pmatrix} 4 \\ -1 \\ 0 \\ 1 \end{pmatrix}$$

Question 4. Find the vector form of the general solution of the linear system $A\boldsymbol{x} = \boldsymbol{b}$, and then use that result to find the vector form of the general solution of $A\boldsymbol{x} = \boldsymbol{0}$.

1.

$$\begin{aligned} x_1 - 3x_3 &= 1 \\ 2x_1 - 6x_2 &= 2 \end{aligned}$$

2.

$$\begin{aligned} x_1 + x_2 + 2x_3 &= 5 \\ x_1 + x_3 &= -2 \\ 2x_1 + x_2 + 3x_3 &= 3 \end{aligned}$$

1. The reduced row echelon form of the augmented matrix is:

$$\left(\begin{array}{ccc|c} 1 & 0 & -3 & 1 \\ 0 & 1 & -1 & 0 \end{array}\right)$$

The general solution of the system is

$$x_1 = 3t + 1, \quad x_2 = t, \quad x_3 = t.$$

A vector form of the general solution of $A\boldsymbol{x} = \boldsymbol{b}$ is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}.$$

A vector form of the general solution of $A\boldsymbol{x} = \boldsymbol{0}$ is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = t \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}.$$

2. The reduced row echelon form of the augmented matrix is:

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & -2 \\ 0 & 1 & 1 & 7 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

The general solution of the system is

$$x_1 = -t - 2, \quad x_2 = 7 - t, \quad x_3 = t.$$

A vector form of the general solution of $A\boldsymbol{x} = \boldsymbol{b}$ is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2 \\ 7 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}.$$

A vector form of the general solution of $A\boldsymbol{x} = \boldsymbol{0}$ is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = t \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}.$$

Question 5. Find bases for the null space and row space of A.

1. $A = \begin{pmatrix} 1 & -1 & 3 \\ 5 & -4 & -4 \\ 7 & -6 & 2 \end{pmatrix}$
2. $A = \begin{pmatrix} 2 & 0 & -1 \\ 4 & 0 & -2 \\ 0 & 0 & 0 \end{pmatrix}$
3. $A = \begin{pmatrix} 1 & 4 & 5 & 2 \\ 2 & 1 & 3 & 0 \\ -1 & 3 & 2 & 2 \end{pmatrix}$
4. $A = \begin{pmatrix} 1 & 4 & 5 & 6 & 9 \\ 3 & -2 & 1 & 4 & -1 \\ -1 & 0 & -1 & -2 & -1 \\ 2 & 3 & 5 & 7 & 8 \end{pmatrix}$

Solution.

1. The reduced row echelon form of the homogeneous system $A\boldsymbol{x} = \mathbf{0}$ is

$$\left(\begin{array}{ccc|c} 1 & 0 & -16 & 0 \\ 0 & 1 & -19 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

A vector form of the solution is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 16t \\ 19t \\ t \end{pmatrix} = t \begin{pmatrix} 16 \\ 19 \\ 1 \end{pmatrix}, \quad t \in \mathbb{R}$$

Thus, a basis for the null space of A is given by

$$\left\{ \begin{pmatrix} 16 \\ 19 \\ 1 \end{pmatrix} \right\}$$

Since the nonzero rows of the reduced row echelon form correspond to a basis for the row space of, a basis for the row space of A is given by

$$\left\{ \begin{pmatrix} 1, & 0, & -16 \end{pmatrix}, \quad \begin{pmatrix} 0, & 1, & -19 \end{pmatrix} \right\}$$

2. The reduced row echelon form of the homogeneous system $A\boldsymbol{x} = \mathbf{0}$ is

$$\left(\begin{array}{ccc|c} 1 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

A vector form of the solution is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{t}{2} \\ s \\ t \end{pmatrix} = t \begin{pmatrix} \frac{1}{2} \\ 0 \\ 1 \end{pmatrix} + s \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad t, s \in \mathbb{R}$$

Thus, a basis for the null space of A is given by

$$\left\{ \begin{pmatrix} \frac{1}{2} \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

A basis for the row space of A is given by

$$\left\{\begin{pmatrix}1, & 0, & -\frac{1}{2}\end{pmatrix}\right\}$$

3. The reduced row echelon form of the homogeneous system $A\boldsymbol{x} = \mathbf{0}$ is

$$\left(\begin{array}{cccc|c} 1 & 0 & 1 & -\frac{2}{7} & 0 \\ 0 & 1 & 1 & \frac{4}{7} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

A vector form of the solution is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} \frac{2t}{7} - s \\ \frac{4t}{7} - s \\ s \\ t \end{pmatrix} = t\begin{pmatrix} \frac{2}{7} \\ \frac{4}{7} \\ 0 \\ 1 \end{pmatrix} + s\begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \qquad t, s \in \mathbb{R}$$

Thus, a basis for the null space of A is given by

$$\left\{\begin{pmatrix} \frac{2}{7} \\ \frac{4}{7} \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix}\right\}$$

A basis for the row space of A is given by

$$\left\{\begin{pmatrix}1, & 0, & 1, & -\frac{2}{7}\end{pmatrix}, \begin{pmatrix}0, & 1, & 1, & \frac{4}{7}\end{pmatrix}\right\}$$

4. The reduced row echelon form of the homogeneous system $A\boldsymbol{x} = \mathbf{0}$ is

$$\left(\begin{array}{ccccc|c} 1 & 0 & 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

A vector form of the solution is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -r - 2s - t \\ -r - s - 2t \\ r \\ s \\ t \end{pmatrix} = r\begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + s\begin{pmatrix} -2 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -1 \\ -2 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \qquad r, s, t \in \mathbb{R}$$

Thus, a basis for the null space of A is given by

$$\left\{\begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -2 \\ 0 \\ 0 \\ 1 \end{pmatrix}\right\}$$

A basis for the row space of A is given by

$$\{(1,\ 0,\ 1,\ 2,\ 1),\quad (0,\ 1,\ 1,\ 1,\ 2)\}$$

Question 6. By inspection, find a basis for the row space and for the column space of the given matrix

1. $A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

2. $A = \begin{pmatrix} 1 & -3 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

3. $A = \begin{pmatrix} 1 & 2 & 4 & 5 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

4. $A = \begin{pmatrix} 1 & 2 & -1 & 5 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 1 & -7 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Solution.

1. A basis for the row space of A is

$$\{(1,\ 0,\ 2),\quad (0,\ 0,\ 1)\}$$

A basis for the column space of A consist of the following vectors

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},\quad \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

2. A basis for the row space of A is

$$\{(1,\ -3,\ 0,\ 0),\quad (0,\ 1,\ 0,\ 0)\}$$

A basis for the column space of A consist of the following vectors

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix},\quad \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

3. A basis for the row space of A is

$$\{(1,\ 2,\ 4,\ 5),\quad (0,\ 1,\ -3,\ 0),\quad (0,\ 0,\ 1,\ -3),\quad (0,\ 0,\ 0,\ 1)\}$$

A basis for the column space of A consist of the following vectors

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},\quad \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix},\quad \begin{pmatrix} 4 \\ -3 \\ 1 \\ 0 \\ 0 \end{pmatrix},\quad \begin{pmatrix} 5 \\ 0 \\ -3 \\ 1 \\ 0 \end{pmatrix}$$

4. A basis for the row space of A is

$$\{(1,\ 2,\ -1,\ 5),\quad (0,\ 1,\ 4,\ 3),\quad (0,\ 0,\ 1,\ -7),\quad (0,\ 0,\ 0,\ 1)\}$$

A basis for the column space of A consist of the following vectors

$$\begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}2\\1\\0\\0\end{pmatrix}, \begin{pmatrix}-1\\4\\1\\0\end{pmatrix}, \begin{pmatrix}5\\3\\-7\\1\end{pmatrix}$$

Question 7. Construct a matrix whose null space consists of all linear combinations of the vectors

$$\boldsymbol{v}_1 = \begin{pmatrix}1\\-1\\3\\2\end{pmatrix}, \quad \text{and} \quad \boldsymbol{v}_2 = \begin{pmatrix}2\\0\\-2\\4\end{pmatrix}$$

Solution. Since the null space of A and the row space of A are orthogonal complements, it suffices to find a basis for the orthogonal complement of span $(\{\boldsymbol{v}_1, \boldsymbol{v}_2\})$.

Equivalently, we seek a basis for the space of vectors $\boldsymbol{x}$, that satisfy the homogeneous system

$$\boldsymbol{v}_1^\top \cdot \boldsymbol{x} = \boldsymbol{0}, \quad \boldsymbol{v}_2^\top \cdot \boldsymbol{x} = \boldsymbol{0}.$$

Thus, we aim to find a basis for the solution space of the system $A\boldsymbol{x} = \boldsymbol{0}$, where the rows of A are given by $\boldsymbol{v}_1^\top$ and $\boldsymbol{v}_2^\top$. The reduced row echelon form of the augmented matrix is

$$\left(\begin{array}{cccc|c}1 & 0 & -1 & 2 & 0\\0 & 1 & -4 & 0 & 0\end{array}\right)$$

Expressing the general solution in vector form, we obtain

$$\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix} = \begin{pmatrix}-2t+s\\4s\\s\\t\end{pmatrix} = t\begin{pmatrix}-2\\0\\0\\1\end{pmatrix} + s\begin{pmatrix}1\\4\\1\\0\end{pmatrix}, \qquad t, s \in \mathbb{R}.$$

Thus a basis for the solution space of $A\boldsymbol{x} = \boldsymbol{0}$ consists of vectors

$$\begin{pmatrix}-2\\0\\0\\1\end{pmatrix}, \begin{pmatrix}1\\4\\1\\0\end{pmatrix}.$$

Consequently, a possible choice for the matrix A is given by

$$\begin{pmatrix}-2 & 0 & 0 & 1\\1 & 4 & 1 & 0\end{pmatrix}$$

Question 8. Find the rank and nullity of the matrix A by reducing it to row echelon form.

1. $A = \begin{pmatrix} 1 & 2 & -1 & 1 \\ 2 & 4 & -2 & 2 \\ 3 & 6 & -3 & 3 \\ 4 & 8 & -4 & 4 \end{pmatrix}$

2. $A = \begin{pmatrix} 1 & -2 & 2 & 3 & -1 \\ -3 & 6 & -1 & 1 & -7 \\ 2 & -4 & 5 & 8 & -4 \end{pmatrix}$

3. $A = \begin{pmatrix} 1 & 0 & -2 & 1 & 0 \\ 0 & -1 & -3 & 1 & 3 \\ -2 & -1 & 1 & -1 & 3 \\ 0 & 1 & 3 & 0 & -4 \end{pmatrix}$

4. $A = \begin{pmatrix} 1 & 3 & 1 & 3 \\ 0 & 1 & 1 & 0 \\ -3 & 0 & 6 & -1 \\ 3 & 4 & -2 & 1 \\ 2 & 0 & -4 & -2 \end{pmatrix}$

1. The reduced row echelon form of A is
$$\begin{pmatrix} 1 & 2 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$
rank $(A) = 1$, nullity $(A) = 3$.

2. The reduced row echelon form of A is
$$\begin{pmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$
rank $(A) = 2$, nullity $(A) = 3$.

3. The reduced row echelon form of A is
$$\begin{pmatrix} 1 & 0 & -2 & 0 & 1 \\ 0 & 1 & 3 & 0 & -4 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$
rank $(A) = 3$, nullity $(A) = 2$.

4. The reduced row echelon form of A is
$$\begin{pmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$
rank $(A) = 3$, nullity $(A) = 1$.

Question 9. The matrix R is the reduced row echelon form of the matrix A.

(a) By inspection of the matrix R, find the rank and nullity of A.

(b) Find the number of leading variables and the number of parameters in the general solution of $A\boldsymbol{x} = \mathbf{0}$ without solving the system.

1. $A = \begin{pmatrix} 2 & -1 & -3 \\ -1 & 2 & -3 \\ 1 & 1 & 4 \end{pmatrix}$ $\quad R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

2. $A = \begin{pmatrix} 2 & -1 & -3 \\ -1 & 2 & -3 \\ 1 & 1 & -6 \end{pmatrix}$ $\quad R = \begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{pmatrix}$

3. $A = \begin{pmatrix} 2 & -1 & -3 \\ -2 & 1 & 3 \\ -4 & 2 & 6 \end{pmatrix}$ $\quad R = \begin{pmatrix} 1 & -\frac{1}{2} & -\frac{3}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

4. $A = \begin{pmatrix} 0 & 2 & 2 & 4 \\ 1 & 0 & -1 & -3 \\ 2 & 3 & 1 & 1 \\ -2 & 1 & 3 & -2 \end{pmatrix}$ $\quad R = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

Solution. Recall that we have discussed in the lecture

- rank (A) =the number of leading variables in the general solution of $A\boldsymbol{x} = \mathbf{0}$
- nullity (A) =the number of parameters in the general solution of $A\boldsymbol{x} = \mathbf{0}$

1. rank $(A) = 3$, nullity $(A) = 0$
2. rank $(A) = 2$, nullity $(A) = 1$
3. rank $(A) = 1$, nullity $(A) = 2$
4. rank $(A) = 3$, nullity $(A) = 1$

Question 10. Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be the linear transformation defined by the formula

$$T \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 + 3x_2 \\ x_1 - x_2 \\ x_1 \end{pmatrix}.$$

1. Find the rank of the standard matrix for T.
2. Find the nullity of the standard matrix for T.

Solution. The standard matrix for T is given by

$$A = \begin{pmatrix} 1 & 3 \\ 1 & -1 \\ 1 & 0 \end{pmatrix}$$

The reduced row echelon form of A is

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

We have $\text{rank}\,(A) = 2$, $\text{nullity}\,(A) = 0$.

Question 11. Let $T : \mathbb{R}^5 \to \mathbb{R}^3$ be the linear transformation defined by the formula

$$T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ x_2 + x_3 + x_4 \\ x_4 + x_5 \end{pmatrix}$$

1. Find the rank of the standard matrix for T.
2. Find the nullity of the standard matrix for T.

Solution. The standard matrix for T is given by

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

The reduced row echelon form of A is

$$\begin{pmatrix} 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

We have $\text{rank}\,(A) = 3$, $\text{nullity}\,(A) = 2$.

Question 12. Discuss how the rank of A varies with t.

1. $A = \begin{pmatrix} 1 & 1 & t \\ 1 & 1 & 1 \\ t & 1 & 1 \end{pmatrix}$
2. $A = \begin{pmatrix} t & 3 & -1 \\ 3 & 6 & -2 \\ -1 & -3 & t \end{pmatrix}$

Solution.

1. If $t = 1$, $\text{rank}\,(A) = 1$. If $t \neq 1$, apply row reduction

$$A \xrightarrow[R_3 \to R_3 - tR_1]{R_2 \to R_2 - R_1} \begin{pmatrix} 1 & 1 & t \\ 0 & 0 & 1-t \\ 0 & 1-t & 1-t^2 \end{pmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{pmatrix} 1 & 1 & t \\ 0 & 1-t & 1-t^2 \\ 0 & 0 & 1-t \end{pmatrix} \xrightarrow[R_3 \to \frac{1}{t-1} R_3]{R_2 \to \frac{1}{t-1} R_2} \begin{pmatrix} 1 & 1 & t \\ 0 & 1 & t+1 \\ 0 & 0 & 1 \end{pmatrix}$$

   Thus, when $t \neq 1$, $\text{rank}\,(A) = 3$

Question 13. Are there values of r and s for which

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r-2 & 2 \\ 0 & s-1 & r+2 \\ 0 & 0 & 3 \end{pmatrix}$$

has rank 1? Has rank 2? If so, find those values.

Solution. Since the first and fourth rows of A are linearly independent, the rank of A must be at least 2. Therefore, there do not exist values of r and s for which $\operatorname{rank}(A) = 1$.

For A to have rank 2, the second and third rows must be contained within the vector space

$$(\operatorname{span}(\{(1, \ 0, \ 0), (0, \ 0, \ 3)\}))$$

This condition requires that the second and third rows be expressible as linear combinations of these two basis vectors. That is, there must exist scalars $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$ such that

$$\alpha_1 (1, \ 0, \ 0) + \alpha_2 (0, \ 0, \ 3) = (0, \ r-2, \ 2), \quad \beta_1 (1, \ 0, \ 0) + \beta_2 (0, \ 0, \ 3) = (0, \ s-1, \ r+2)$$

By comparing components, we obtain the system of equations:

$$\alpha_1 = 0, \quad r-2=0, \quad 3\alpha_2 = 2, \quad \beta_1 = 0, \quad s-1=0, \quad 3\beta_2 = r+2.$$

Solving for r and s we find

$$r = 2, \quad s = 1.$$

Thus, we conclude that $\operatorname{rank}(A) = 2$ iff $r = 2, s = 1$.

Question 14.

1. If A is a 3×5 matrix, then the rank of A is at most _____.
2. If A is a 3×5 matrix, then the nullity of A is at most _____.
3. If A is a 3×5 matrix, then the rank of $A^\top$ is at most _____.
4. If A is a 5×3 matrix, then the nullity of $A^\top$ is at most _____.

Solution.

1. If A is a 3×5 matrix, then the rank of A is at most **5**.
2. If A is a 3×5 matrix, then the nullity of A is at most **5**.
3. If A is a 3×5 matrix, then the rank of $A^\top$ is at most **3**.
4. If A is a 5×3 matrix, then the nullity of $A^\top$ is at most **5**.

Question 15.

1. If A is a 3×5 matrix, then the number of leading 1's in the reduced row echelon form of A is at most _____.
2. If A is a 3×5 matrix, then the number of parameters in the general solution of $A\boldsymbol{x} = \mathbf{0}$ is at most _____.
3. If A is a 5×3 matrix, then the number of leading 1's in the reduced row echelon form of A is at most _____.
4. If A is a 5×3 matrix, then the number of parameters in the general solution of $A\boldsymbol{x} = \mathbf{0}$ is at most _____.

Solution.

1. If A is a 3×5 matrix, then the number of leading 1's in the reduced row echelon form of A is at most **5**.
2. If A is a 3×5 matrix, then the number of parameters in the general solution of $A\boldsymbol{x} = \mathbf{0}$ is at most **5**.
3. If A is a 5×3 matrix, then the number of leading 1's in the reduced row echelon form of A is at most **3**.
4. If A is a 5×3 matrix, then the number of parameters in the general solution of $A\boldsymbol{x} = \mathbf{0}$ is at most **3**.

Question 16. Let

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}.$$

Show that A has rank 2 if and only if one or more of the following determinants is nonzero:

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, \quad \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}, \quad \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}.$$

Solution. A has rank 2 if and only if its two rows are linearly independent. That is, the second row of A cannot be written as a scalar multiple of the first row. This means the following system of equations in the unknown x has no solutions

$$x\begin{pmatrix} a_{11}, & a_{12}, & a_{13} \end{pmatrix} = \begin{pmatrix} a_{21}, & a_{22}, & a_{23} \end{pmatrix}$$

Expanding component-wise, this is equivalent to the system

$$\begin{aligned} a_{11}x &= a_{21} \\ a_{12}x &= a_{22} \\ a_{13}x &= a_{23} \end{aligned}$$

Then the system has no solutions if there does not exist a single scalar x satisfying all three equations simultaneously. Note that at least one of a_{11}, a_{12}, a_{13} is nonzero. The system having no solutions is then equivalent to requiring that at least one of the following inequalities hold:

$$a_{22}a_{11} \neq a_{21}a_{12}, \quad a_{11}a_{23} \neq a_{21}a_{13}, \quad a_{12}a_{23} \neq a_{22}a_{13}.$$

Rewriting in determinant form, this means that at least one of the following determinants must be nonzero:

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, \quad \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}, \quad \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}.$$

Question 17. Determine whether the matrix operator $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by the equations is bijective.

1.

$$\begin{aligned} w_1 &= x_1 + 2x_2 \\ w_2 &= -x_1 + x_2 \end{aligned}$$

2.

$$\begin{aligned} w_1 &= 4x_1 - 6x_2 \\ w_2 &= -2x_1 + 3x_2 \end{aligned}$$

3.

$$\begin{aligned} w_1 &= x_1 - 2x_2 + 2x_3 \\ w_2 &= 2x_1 + x_2 + x_3 \\ w_3 &= x_1 + x_2 \end{aligned}$$

4.

$$\begin{aligned} w_1 &= x_1 - 3x_2 + 4x_3 \\ w_2 &= -x_1 + x_2 + x_3 \\ w_3 &= -2x_2 + 5x_3 \end{aligned}$$

Solution.

1. The standard matrix for T is
$$A = \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix}$$
since $\det(A) = 3 \neq 0$, T is bijective.

2. The standard matrix for T is
$$A = \begin{pmatrix} 4 & -6 \\ -2 & 3 \end{pmatrix}$$
since $\det(A) = 0$, T is not bijective.

3. The standard matrix for T is
$$A = \begin{pmatrix} 1 & -2 & 2 \\ 2 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$
since $\det(A) = -1 \neq 0$, T is bijective.

4. The standard matrix for T is
$$A = \begin{pmatrix} 1 & -3 & 4 \\ -1 & 1 & 1 \\ 0 & -2 & 5 \end{pmatrix}$$
since $\det(A) = 0$, T is not bijective.

Question 18. Determine whether multiplication by A is an injective matrix transformation.

1. $A = \begin{pmatrix} 1 & -1 \\ 2 & 0 \\ 3 & -4 \end{pmatrix}$

2. $A = \begin{pmatrix} 1 & 2 & 3 \\ -1 & 0 & -4 \end{pmatrix}$

3. $A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix}$

4. $A = \begin{pmatrix} 4 & 3 \\ 1 & 1 \end{pmatrix}$

Solution. The matrix transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ defined by multiplication by a matrix A is injective if and only if for all distinct vectors $\boldsymbol{v}_1, \boldsymbol{v}_2 \in \mathbb{R}^n$, we have

$$A\boldsymbol{v}_1 \neq A\boldsymbol{v}_2.$$

This condition is equivalent to requiring that the only solution to

$$A\boldsymbol{v}_1 = A\boldsymbol{v}_2$$

is when $\boldsymbol{v}_1 = \boldsymbol{v}_2$. Subtracting both sides, we obtain

$$A(\boldsymbol{v}_1 - \boldsymbol{v}_2) = \mathbf{0}.$$

Setting $\boldsymbol{x} = \boldsymbol{v}_1 - \boldsymbol{v}_2$, we conclude that A is injective if and only if the homogeneous equation

$$A\boldsymbol{x} = \mathbf{0}$$

has only the trivial solution $\boldsymbol{x} = \mathbf{0}$. In other words,

$$\text{nullity}(A) = 0.$$

Thus, a matrix transformation given by multiplication by A is injective if and only if $\text{nullity}(A) = 0$.

1. The reduced row echelon form of A is

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

nullity $(A) = 0$ and the matrix transformation is injective.

2. The reduced row echelon form of A is

$$\begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & -\frac{1}{2} \end{pmatrix}$$

nullity $(A) = 1$ and the matrix transformation is not injective.

3. The reduced row echelon form of A is

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

nullity $(A) = 1$ and the matrix transformation is not injective.

4. The reduced row echelon form of A is

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

nullity $(A) = 0$ and the matrix transformation is injective.

Question 19. Let T_A be multiplication by the matrix A. Find:

(a) a basis for the range of T_A.

(b) a basis for the kernel of T_A.

(d) the rank and nullity of A.

1. $A = \begin{pmatrix} 1 & -1 & 3 \\ 5 & 6 & -4 \\ 7 & 4 & 2 \end{pmatrix}$

2. $A = \begin{pmatrix} 2 & 0 & -1 \\ 4 & 0 & -2 \\ 20 & 0 & 0 \end{pmatrix}$

Solution. Recall that the range of T_A, denoted $\mathrm{R}(T_A)$, is the column space of A.

1. The reduced row echelon form of A is

$$R = \begin{pmatrix} 1 & 0 & \frac{14}{11} \\ 0 & 1 & -\frac{19}{11} \\ 0 & 0 & 0 \end{pmatrix}.$$

Observing R, we note that the first two columns are pivot columns, indicating that they form a basis for the column space of R. Consequently, the first two columns of A form a basis for the column space of A.

(a) A basis for $\mathrm{R}(T_A)$ consists of the following vectors:

$$\begin{pmatrix} 1 \\ 5 \\ 7 \end{pmatrix}, \quad \begin{pmatrix} -1 \\ 6 \\ 4 \end{pmatrix}$$

(b) From R, we can deduce that a vector form for the general solution of the homogeneous system $A\boldsymbol{x} = \boldsymbol{b}$ is

$$\begin{pmatrix} -\frac{14t}{11} \\ \frac{19t}{11} \\ t \end{pmatrix} = t \begin{pmatrix} -\frac{14}{11} \\ \frac{19}{11} \\ 1 \end{pmatrix}$$

Thus a basis for the kernel of T_A consists of the vector

$$\begin{pmatrix} -\frac{14}{11} \\ \frac{19}{11} \\ 1 \end{pmatrix}$$

(c) $\operatorname{rank}(A) = 2$, $\operatorname{nullity}(A) = 1$.

2. The reduced row echelon form of A is

$$R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Observing R, we note that the first and the last columns are pivot columns, indicating that they form a basis for the column space of R. Consequently, the first and the last columns of A form a basis for the column space of A.

(a) A basis for $\mathrm{R}(T_A)$ consists of the following vectors:

$$\begin{pmatrix} 2 \\ 4 \\ 20 \end{pmatrix}, \quad \begin{pmatrix} -1 \\ -2 \\ 0 \end{pmatrix}$$

(b) From R, we can deduce that a vector form for the general solution of the homogeneous system $A\boldsymbol{x} = \boldsymbol{b}$ is

$$\begin{pmatrix} 0 \\ t \\ 0 \end{pmatrix} = t \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

Thus a basis for the kernel of T_A consists of the vector

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

(c) $\operatorname{rank}(A) = 2$, $\operatorname{nullity}(A) = 1$.

Question 20. Let $T_A : \mathbb{R}^4 \to \mathbb{R}^3$ be multiplication by A. Find:

(a) a basis for the kernel of T_A.

(b) a basis for the range of T_A that consists of column vectors of A.

1. $A = \begin{pmatrix} 1 & 2 & -1 & -2 \\ -3 & 1 & 3 & 4 \\ -3 & 8 & 4 & 2 \end{pmatrix}$

2. $A = \begin{pmatrix} 1 & 1 & 0 & 1 \\ -2 & 4 & 2 & 2 \\ -1 & 8 & 3 & 5 \end{pmatrix}$

Solution.

1. The reduced row echelon form of A is

$$R = \begin{pmatrix} 1 & 0 & 0 & -\frac{10}{7} \\ 0 & 1 & 0 & -\frac{2}{7} \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

(a) From R, we can deduce that a vector form for the general solution of the homogeneous system $A\boldsymbol{x} = \boldsymbol{b}$ is

$$\begin{pmatrix} \frac{10t}{7} \\ \frac{2t}{7} \\ 0 \\ t \end{pmatrix} = t \begin{pmatrix} \frac{10}{7} \\ \frac{2}{7} \\ 0 \\ 1 \end{pmatrix}$$

Thus a basis for the kernel of T_A consists of the vector

$$\begin{pmatrix} \frac{10}{7} \\ \frac{2}{7} \\ 0 \\ 1 \end{pmatrix}$$

(b) Observing R, we note that the first three columns are pivot columns, indicating that they form a basis for the column space of R. Consequently, the first three columns of A form a basis for the column space of A. A basis for $\mathrm{R}(T_A)$ consists of the following vectors:

$$\begin{pmatrix} 1 \\ -3 \\ -3 \end{pmatrix}, \quad \begin{pmatrix} 2 \\ 1 \\ 8 \end{pmatrix}, \quad \begin{pmatrix} -1 \\ 3 \\ 4 \end{pmatrix}$$

2. The reduced row echelon form of A is

$$R = \begin{pmatrix} 1 & 0 & -\frac{1}{3} & \frac{1}{3} \\ 0 & 1 & \frac{1}{3} & \frac{2}{3} \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

(a) From R, we can deduce that a vector form for the general solution of the homogeneous system $A\boldsymbol{x} = \boldsymbol{b}$ is

$$\begin{pmatrix} \frac{s}{3} - \frac{t}{3} \\ -\frac{s}{3} - \frac{2t}{3} \\ s \\ t \end{pmatrix} = s \begin{pmatrix} \frac{1}{3} \\ -\frac{1}{3} \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -\frac{1}{3} \\ -\frac{2}{3} \\ 0 \\ 1 \end{pmatrix}$$

Thus a basis for the kernel of T_A consists of the vectors

$$\begin{pmatrix} \frac{1}{3} \\ -\frac{1}{3} \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} -\frac{1}{3} \\ -\frac{2}{3} \\ 0 \\ 1 \end{pmatrix}$$

(b) Observing R, we note that the first two columns are pivot columns, indicating that they form a basis for the column space of R. Consequently, the first two columns of A form a basis for the column space of A. A basis for $\mathrm{R}(T_A)$ consists of the following vectors:

$$\begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 4 \\ 8 \end{pmatrix}.$$

Question 21. Let A be an $n \times n$ matrix such that $\det(A) = 0$, and let $T : \mathbb{R}^n \to \mathbb{R}^n$ be multiplication by A.

1. What can you say about the range of the matrix operator T? Give an example that illustrates your conclusion.
2. What can you say about the number of vectors that T maps into $\mathbf{0}$?

What if $\det(A) \neq 0$?

Solution.

1. Since $\det(A) = 0$, the range of T is a subspace of $\mathbb{R}^n$. T is not surjective, and there exist vectors in $\mathbb{R}^n$ that are not in the image of A.

   Example: Consider the matrix
   $$A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}.$$
   The determinant of A is 0. The column space of A is spanned by a single vector $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$. This means that A maps all vectors in $\mathbb{R}^2$ to a one-dimensional subspace of $\mathbb{R}^2$, confirming that T is not surjective.

2. Since $\det(A) = 0$, the null space of A is nontrivial. This means that there exist nonzero vectors $\boldsymbol{x} \neq \mathbf{0}$ such that
   $$A\boldsymbol{x} = \mathbf{0}.$$
   In other words, T is not injective, and there are infinitely many vectors that T maps to $\mathbf{0}$.

   Example: For the matrix A given above, we solve $A\boldsymbol{x} = \mathbf{0}$ and we get a basis for the kernel of T that consists of the following vector
   $$\begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$
   Any scalar multiple of this vector is mapped to $\mathbf{0}$ by T.

If $\det(A) \neq 0$, then $\mathrm{R}(T) = \mathbb{R}^n$ and only the zero vector $\mathbf{0}$ is mapped to $\mathbf{0}$ by T.

Question 22. Confirm by multiplication that $\boldsymbol{x}$ is an eigenvector of A, and find the corresponding eigenvalue.

1. $A = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}, \quad \boldsymbol{x} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

2. $A = \begin{pmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 0 & 4 \end{pmatrix}, \quad \boldsymbol{x} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$

3. $A = \begin{pmatrix} 5 & -1 \\ 1 & 3 \end{pmatrix}, \quad \boldsymbol{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

4. $A = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}, \quad \boldsymbol{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

Solution.

1. $A\boldsymbol{x} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$, the corresponding eigenvalue is -1.
2. $A\boldsymbol{x} = \begin{pmatrix} 5 \\ 10 \\ 5 \end{pmatrix}$, the corresponding eigenvalue is 5.
3. $A\boldsymbol{x} = \begin{pmatrix} 4 \\ 4 \end{pmatrix}$, the corresponding eigenvalue is 4.
4. $A\boldsymbol{x} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, the corresponding eigenvalue is 0.

Question 23. Find the characteristic equation, the eigenvalues, and the bases for the eigenspaces of the matrix A.

1. $A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$

2. $A = \begin{pmatrix} -2 & -7 \\ 1 & 2 \end{pmatrix}$

3. $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

4. $A = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$

5. $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$

6. $A = \begin{pmatrix} 2 & -3 \\ 0 & 2 \end{pmatrix}$

7. $A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

8. $A = \begin{pmatrix} 1 & 2 \\ -2 & -1 \end{pmatrix}$

9. $A = \begin{pmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$

10. $A = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & 4 \end{pmatrix}$

11. $A = \begin{pmatrix} 6 & 3 & -8 \\ 0 & -2 & 0 \\ 1 & 0 & -3 \end{pmatrix}$

12. $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

13. $A = \begin{pmatrix} 4 & 0 & -1 \\ 0 & 3 & 0 \\ 1 & 0 & 2 \end{pmatrix}$

14. $A = \begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix}$

Solution.

1. The characteristic equation of A is given by

$$\det(\lambda I - A) = \begin{vmatrix} \lambda - 1 & -4 \\ -2 & \lambda - 3 \end{vmatrix} = \lambda^2 - 4\lambda - 5 = 0$$

Solving for λ, we obtain the eigenvalues:

$$\lambda_1 = 5, \quad \lambda_2 = -1$$

To determine a basis for the eigenspace corresponding to $\lambda_1 = 5$, we solve the homogeneous system

$$(5I - A)\boldsymbol{x} = \mathbf{0},$$

where

$$5I - A = \begin{pmatrix} 4 & -4 \\ -2 & 2 \end{pmatrix}.$$

Reducing the system, we find the general solution:

$$\boldsymbol{x} = t \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Thus, a basis for the eigenspace corresponding to $\lambda_1 = 5$ is

$$\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}.$$

Similarly, to determine a basis for the eigenspace corresponding to $\lambda_2 = -1$, we solve the homogeneous system

$$\begin{pmatrix} -2 & -4 \\ -2 & -4 \end{pmatrix} \boldsymbol{x} = \mathbf{0}.$$

We obtain the general solution:

$$\boldsymbol{x} = t \begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

Thus, a basis for the eigenspace corresponding to $\lambda_2 = -1$ is

$$\left\{ \begin{pmatrix} -2 \\ 1 \end{pmatrix} \right\}.$$

2. Characteristic equation: $\lambda^2 + 3 = 0$

$$\lambda_1 = \sqrt{3}i, \text{ basis vector: } \begin{pmatrix} \dfrac{7}{-2 - \sqrt{3}i} \\ 1 \end{pmatrix}$$

$$\lambda_2 = -\sqrt{3}i, \text{ basis vector: } \begin{pmatrix} \dfrac{7}{-2 + \sqrt{3}i} \\ 1 \end{pmatrix}$$

3. Characteristic equation: $\lambda^2 - 2\lambda + 1 = 0$

$$\lambda = 1, \text{ basis vectors: } \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

4. Characteristic equation: $\lambda^2 - 2\lambda + 1 = 0$

$$\lambda = 1, \text{ basis vector: } \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

5. Characteristic equation: $\lambda^2 - 4\lambda + 3 = 0$

$$\lambda_1 = 1, \text{ basis vector: } \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\lambda_2 = 3, \text{ basis vector: } \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

6. Characteristic equation: $\lambda^2 - 4\lambda + 4 = 0$

$$\lambda = 2, \text{ basis vector: } \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

7. Characteristic equation: $\lambda^2 - 4\lambda + 4 = 0$

$$\lambda = 2, \text{ basis vector: } \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

8. Characteristic equation: $\lambda^2 + 3 = 0$

$$\lambda_1 = \sqrt{3}i, \text{ basis vector: } \begin{pmatrix} -\dfrac{2}{1-\sqrt{3}i} \\ 1 \end{pmatrix}$$

$$\lambda_2 = -\sqrt{3}i, \text{ basis vector: } \begin{pmatrix} -\dfrac{2}{1+\sqrt{3}i} \\ 1 \end{pmatrix}$$

9. Characteristic equation: $-\lambda^3 + 6\lambda^2 - 11\lambda + 6 = 0$

$$\lambda_1 = 1, \text{ basis vector: } \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\lambda_2 = 2, \text{ basis vector: } \begin{pmatrix} -\dfrac{1}{2} \\ 1 \\ 1 \end{pmatrix}$$

$$\lambda_3 = 3, \text{ basis vector: } \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

10. Characteristic equation: $-\lambda^3 + 5\lambda^2 = 0$

$$\lambda_1 = 0, \text{ basis vectors: } \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$$

$$\lambda_2 = 5, \text{ basis vector: } \begin{pmatrix} -\dfrac{1}{2} \\ 0 \\ 1 \end{pmatrix}$$

11. Characteristic equation: $-\lambda^3 + \lambda^2 + 16\lambda + 20 = 0$

$$\lambda_1 = 5, \text{ basis vector: } \begin{pmatrix} 8 \\ 0 \\ 1 \end{pmatrix}$$

$$\lambda_2 = -2, \text{ basis vector: } \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

12. Characteristic equation: $-\lambda^3 + 3\lambda + 2 = 0$

$$\lambda_1 = 2, \text{ basis vector: } \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\lambda_2 = -1, \text{ basis vectors: } \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

13. Characteristic equation: $-\lambda^3 + 9\lambda^2 - 27\lambda + 27 = 0$

$$\lambda = 3, \text{ basis vector: } \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

14. Characteristic equation: $-\lambda^3 + 12\lambda + 16 = 0$

$$\lambda_1 = -2, \text{ basis vector: } \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 1 \end{pmatrix}$$

$$\lambda_2 = 4, \text{ basis vectors: } \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

Question 24. Find the characteristic equation of the matrix by inspection.

1. $A = \begin{pmatrix} 3 & 0 & 0 \\ -2 & 7 & 0 \\ 4 & 8 & 1 \end{pmatrix}$

2. $A = \begin{pmatrix} 9 & -8 & 6 & 3 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 7 \end{pmatrix}$

Solution.

1. $(\lambda - 3)(\lambda - 7)(\lambda - 1) = 0$
2. $(\lambda - 9)(\lambda + 1)(\lambda - 3)(\lambda - 7) = 0$

Question 25. Find the eigenvalues and the corresponding eigenspaces of the stated matrix operator on $\mathbb{R}^2$. Use geometric reasoning to help finding the answers.

1. Reflection about the line $y = x$.
2. Orthogonal projection onto the x-axis.
3. Rotation about the origin through a positive angle of 90°.
4. Contraction with factor α $(0 < \alpha < 1)$.
5. Shear in the $x-$direction by a factor α $(\alpha \neq 0)$.
6. Reflection about the $y-$axis.
7. Rotation about the origin through a positive angle of 180°.
8. Dilation with factor α $(\alpha > 1)$.
9. Expansion in the $y-$direction with factor α $(\alpha > 1)$.
10. Shear in the $y-$direction by a factor α $(\alpha \neq 0)$.

Solution.

1. The standard matrix for the reflection about the line $y = x$

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

To determine the eigenvectors of A, we note that an eigenvector must remain on the same line after reflection. This occurs only if the vector lies either along the line $y = x$ or along the line perpendicular to $y = x$, which is $y = -x$. Thus, the only possible eigenvectors are those parallel to $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ or $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

Furthermore, since the reflection does not change the magnitude of the vectors, the possible eigenvalues are only $1, -1$.

In conclusion

- The eigenspace corresponding to $\lambda_1 = 1$ consists of all scalar multiples of $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, so a basis for this eigenspace is

$$\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}.$$

- The eigenspace corresponding to $\lambda_2 = -1$ consists of all scalar multiples of $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$, so a basis for this eigenspace is

$$\left\{ \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}.$$

2. The standard matrix is

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

An eigenvector must remain on the same line after projection – they are either on the $x-$axis or $y-$axis

$$\lambda_1 = 0, \text{ basis vector: } \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\lambda_2 = 1, \text{ basis vector: } \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

3. The standard matrix is

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\lambda_1 = i, \text{ basis vector: } \begin{pmatrix} i \\ 1 \end{pmatrix}$$

$$\lambda_2 = -i, \text{ basis vector: } \begin{pmatrix} -i \\ 1 \end{pmatrix}$$

4. The standard matrix is

$$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}$$

Any vector in $\mathbb{R}^2$ serves as an eigenvector since contraction scales all vectors without altering their directions.

$$\lambda = \alpha, \text{ basis vectors: } \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

5. The standard matrix is

$$\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$$

The only vectors that remain collinear after a shearing transformation in the $x-$direction are those that initially lie along the $x-$axis.

$$\lambda = 1, \text{ basis vector: } \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

6. The standard matrix is

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

The vectors that remain collinear after reflection about the $y-$axis are those that lie either on the $y-$axis or the $x-$axis.

$$\lambda = 1, \text{ basis vector: } \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\lambda = -1, \text{ basis vector: } \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

7. The standard matrix is

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

All vectors in $\mathbb{R}^2$ remain collinear after a $180°$ rotation, with their directions reversed.

$$\lambda = -1, \text{ basis vectors: } \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

8. The standard matrix is
$$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}$$
Any vector in $\mathbb{R}^2$ serves as an eigenvector since dilation scales all vectors without altering their directions.
$$\lambda = \alpha, \text{ basis vectors: } \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

9. The standard matrix is
$$\begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}$$
The eigenvectors consist of those lying on the $x-$axis, which remain unchanged, and those on the $y-$axis, which are scaled by a factor of α.
$$\lambda_1 = 1, \text{ basis vector: } \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
$$\lambda_2 = \alpha, \text{ basis vector: } \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

10. The standard matrix is
$$\begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix}$$
The eigenvectors consist of those lie on the $y-$axis that remain unchanged.
$$\lambda = 1, \text{ basis vectors: } \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Question 26. Find the eigenvalues and the corresponding eigenspaces of the stated matrix operator on $\mathbb{R}^3$. Use geometric reasoning to find the answers.

1. Reflection about the xy-plane.
2. Orthogonal projection onto the xz-plane.
3. Counterclockwise rotation about the positive x-axis through an angle of 90°.
4. Contraction with factor α $(0 \leq \alpha < 1)$.
5. Reflection about the xz-plane.
6. Orthogonal projection onto the yz-plane.
7. Counterclockwise rotation about the positive y-axis through an angle of 180°.
8. Dilation with factor α $(\alpha > 1)$.

Solution.

1. The standard matrix is
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$
The eigenvectors consist of those lying on the $xy-$plane, which remain unchanged, and those on the $z-$axis, whose directions are reversed.
$$\lambda_1 = 1, \text{ basis vectors: } \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$
$$\lambda_2 = -1, \text{ basis vectors: } \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

2. The standard matrix is
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
The eigenvectors consist of those lying on the $y-$axis, which are mapped to $\mathbf{0}$ and those on the $xz-$plane which remain unchanged.
$$\lambda_1 = 0, \text{ basis vector: } \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$
$$\lambda_2 = 1, \text{ basis vectors: } \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

3. The standard matrix is
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$
$$\lambda_1 = 1, \text{ basis vector: } \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$
$$\lambda_2 = i, \text{ basis vector: } \begin{pmatrix} 0 \\ i \\ 1 \end{pmatrix}$$
$$\lambda_3 = -i, \text{ basis vector: } \begin{pmatrix} 0 \\ -i \\ 1 \end{pmatrix}$$

4. The standard matrix is
$$\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{pmatrix}$$
Any vector in $\mathbb{R}^2$ serves as an eigenvector since contraction scales all vectors without altering their directions.

$$\lambda = \alpha, \text{ basis vectors: } \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}$$

5. The standard matrix is

$$\begin{pmatrix}1 & 0 & 0\\0 & -1 & 0\\0 & 0 & 1\end{pmatrix}$$

The eigenvectors consist of those lying on the $xz-$plane, which remain unchanged, and those on the $y-$axis, whose directions are reversed.

$$\lambda_1 = 1, \text{ basis vectors: } \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$\lambda_2 = -1, \text{ basis vectors: } \begin{pmatrix}0\\1\\0\end{pmatrix}$$

6. The standard matrix is

$$\begin{pmatrix}0 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}$$

The eigenvectors consist of those lying on the $x-$axis, which are mapped to $\mathbf{0}$ and those on the $yz-$plane which remain unchanged.

$$\lambda_1 = 0, \text{ basis vector: } \begin{pmatrix}1\\0\\0\end{pmatrix}$$

$$\lambda_2 = 1, \text{ basis vectors: } \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}$$

7. The standard matrix is

$$\begin{pmatrix}-1 & 0 & 0\\0 & 1 & 0\\0 & 0 & -1\end{pmatrix}$$

Vectors along the $y-$axis remain unchanged, while those in the $xz-$plane have their directions reversed after the rotation.

$$\lambda_1 = 1, \text{ basis vector: } \begin{pmatrix}0\\1\\0\end{pmatrix}$$

$$\lambda_2 = -1, \text{ basis vector: } \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}$$

8. The standard matrix is

$$\begin{pmatrix}\alpha & 0 & 0\\0 & \alpha & 0\\0 & 0 & \alpha\end{pmatrix}$$

Any vector in $\mathbb{R}^2$ serves as an eigenvector since dilation scales all vectors without altering their directions.

$$\lambda = \alpha, \text{ basis vectors: } \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}$$

Question 27. Find $\det(A)$ given that A has $p(\lambda)$ as its characteristic polynomial.

1. $p(\lambda) = \lambda^3 - 2\lambda^2 + \lambda + 5$
2. $p(\lambda) = \lambda^4 - \lambda^3 + 7$

Solution.

1. We have
$$\det(\lambda I - A) = \lambda^3 - 2\lambda^2 + \lambda + 5$$
Setting $\lambda = 0$, we obtain
$$\det(-A) = 5.$$
Since the highest power of λ in the characteristic equation corresponds to the size of A, we conclude that $A \in \mathcal{M}_{3\times 3}$.
Using the determinant property $\det(-A) = (-1)^n \det(A)$ for an $n \times n$ matrix, we have
$$\det(-A) = (-1)^3 \det(A) = -\det(A).$$
Substituting $\det(-A) = 5$, we solve for $\det(A)$:
$$-\det(A) = 5 \Longrightarrow \det(A) = -5.$$
2. $(-1)^4 \det(A) = 7$, $\det(A) = 7$

Question 28. Suppose that the characteristic polynomial of some matrix A is found to be

$$p(\lambda) = (\lambda - 1)(\lambda - 3)^2(\lambda - 4)^3.$$

1. What is the size of A?
2. Is A invertible?
3. How many eigenspaces does A have?

Solution.

1. The size of A is determined by the degree of its characteristic polynomial. Since the polynomial is of degree
$$1 + 2 + 3 = 6,$$
we conclude that A is a 6×6 matrix.
2. A matrix is invertible if and only if zero is not an eigenvalue. Substituting $\lambda = 0$ into the polynomail we get
$$p(0) = (-1) \times 9 \times (-64) \neq 0,$$
thus A is invertible.

3. The number of eigenspaces of A corresponds to the number of distinct eigenvalues. From the characteristic polynomial, we see that A has the three distinct eigenvalues:

$$\lambda_1 = 1, \quad \lambda_2 = 3, \quad \lambda_3 = 4.$$

Therefore, A has three eigenspaces.

Question 29. The eigenvectors that we have been studying are sometimes called *right eigenvectors* to distinguish them from *left eigenvectors*, which are $n \times 1$ column vectors $\boldsymbol{x}$ that satisfy the equation

$$\boldsymbol{x}^\top A = \mu \boldsymbol{x}^\top$$

for some scalar μ. For a given matrix A, how are the right eigenvectors and their corresponding eigenvalues related to the left eigenvectors and their corresponding eigenvalues?

Solution. We begin by considering the transpose of the expression $\boldsymbol{x}^\top A$:

$$(\boldsymbol{x}^\top A)^\top = A^\top \boldsymbol{x}.$$

Additionally, we have

$$(\mu \boldsymbol{x}^\top)^\top = \mu \boldsymbol{x}.$$

Thus, if $\boldsymbol{x}$ is a left eigenvector of A corresponding to eigenvalue μ, we have

$$A^\top \boldsymbol{x} = \mu \boldsymbol{x}.$$

This shows that the left eigenvectors of A are the right eigenvectors of $A^\top$, associated with the same eigenvalue.

Question 30. Prove that the characteristic equation of a 2×2 matrix A can be expressed as

$$\lambda^2 - \operatorname{tr}(A)\lambda + \det(A) = 0,$$

where $\operatorname{tr}(A)$ is the trace of A.

Solution. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a 2×2 matrix. Then

$$\det(\lambda I - A) = \begin{vmatrix} \lambda - a & -b \\ -c & \lambda - d \end{vmatrix} = (\lambda - a)(\lambda - d) - bc = \lambda^2 - (a+d)\lambda + (ad - bc)$$

Since $\operatorname{tr}(A) = a + d$, $\det(A) = ad - bc$. The characteristic equation of A is given by

$$\lambda^2 - \operatorname{tr}(A)\lambda + \det(A) = 0.$$

Question 31. Use the result from the Question 30 to show that if

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

then the solutions of the characteristic equation of A are

$$\lambda = \frac{1}{2}\left((a+d) \pm \sqrt{(a-d)^2 + 4bc}\right).$$

Use this result to show that A has

(a) two distinct real eigenvalues if $(a-d)^2+4bc>0$.

(b) two repeated real eigenvalues if $(a-d)^2+4bc=0$.

(c) complex conjugate eigenvalues if $(a-d)^2+4bc<0$.

Solution. Since the characteristic equation of A is given by

$$\lambda^2-\operatorname{tr}(A)\lambda+\det(A)=0.$$

Using the quadratic formula, the solutions to the characteristic equation are:

$$\begin{aligned}\lambda &= \frac{\operatorname{tr}(A)\pm\sqrt{\operatorname{tr}(A)^2-4\det(A)}}{2}=\frac{1}{2}\left((a+d)\pm\sqrt{(a+d)^2-4(ad-bc)}\right)\\ &= \frac{1}{2}\left((a+d)\pm\sqrt{(a-d)^2+4bc}\right)\end{aligned}$$

Consequently, A has two eigenvalues

$$\lambda_1=\frac{1}{2}\left[(a+d)+\sqrt{(a-d)^2+4bc}\right]$$

and

$$\lambda_2=\frac{1}{2}\left[(a+d)-\sqrt{(a-d)^2+4bc}\right].$$

(a) If the discriminant of the quadratic equation satisfies

$$(a-d)^2+4bc>0,$$

then the square root term is a positive real number. Since we are adding and subtracting a positive quantity to $(a+d)$, we obtain two distinct real eigenvalues λ_1 and λ_2.

(b) If

$$(a-d)^2+4bc=0,$$

the quadratic equation has two repeated roots:

$$\lambda_1=\lambda_2=\frac{a+d}{2}.$$

Thus, A has two repeated real eigenvalues.

(c) If

$$(a-d)^2+4bc<0,$$

then the square root term is an imaginary number. This results in two complex conjugate eigenvalues of the form:

$$\lambda_1=\frac{a+d}{2}+i\frac{\sqrt{|(a-d)^2+4bc|}}{2}.$$

and

$$\lambda_2=\frac{a+d}{2}-i\frac{\sqrt{|(a-d)^2+4bc|}}{2}.$$

Thus, A has complex conjugate eigenvalues.

Question 32. Let A be the matrix in from the Question 31. Show that if $b \neq 0$, then

$$\boldsymbol{x}_1 = \begin{pmatrix} -b \\ a - \lambda_1 \end{pmatrix}, \quad \boldsymbol{x}_2 = \begin{pmatrix} -b \\ a - \lambda_2 \end{pmatrix}$$

are eigenvectors of A that correspond, respectively, to the eigenvalues

$$\lambda_1 = \frac{1}{2}\left[(a+d) + \sqrt{(a-d)^2 + 4bc}\right]$$

and

$$\lambda_2 = \frac{1}{2}\left[(a+d) - \sqrt{(a-d)^2 + 4bc}\right].$$

Solution. Computing $A\boldsymbol{x}_1$:

$$A\boldsymbol{x}_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} -b \\ a - \lambda_1 \end{pmatrix} = \begin{pmatrix} -ab + ba - b\lambda_1 \\ -bc + da - d\lambda_1 \end{pmatrix} = \begin{pmatrix} -b\lambda_1 \\ -bc + da - d\lambda_1 \end{pmatrix}.$$

Using the characteristic equation

$$\lambda^2 - (a+d)\lambda + (ad - bc) = 0,$$

we substitute λ_1:

$$\lambda_1^2 - (a+d)\lambda_1 + (ad - bc) = 0 \Longrightarrow ad - bc = -\lambda_1^2 + (a+d)\lambda_1$$

Then

$$A\boldsymbol{x}_1 = \begin{pmatrix} -b\lambda_1 \\ -\lambda_1^2 + (a+d)\lambda_1 - d\lambda_1 \end{pmatrix} = \lambda_1 \begin{pmatrix} -b \\ a - \lambda_1 \end{pmatrix} = \lambda_1 \boldsymbol{x}_1.$$

This confirms that $\boldsymbol{x}_1$ is an eigenvector of A corresponding to λ_1.

Similarly, for $\boldsymbol{x}_2$, we follow the same process and verify:

$$A\boldsymbol{x}_2 = \lambda_2 \boldsymbol{x}_2.$$

Question 33. Use the result of Question 30 to prove that if

$$p(\lambda) = \lambda^2 + c_1\lambda + c_2$$

is the characteristic polynomial of a 2×2 matrix, then

$$p(A) = A^2 + c_1 A + c_2 I = O.$$

(Stated informally, A satisfies its characteristic equation. This result is true as well for $n \times n$ matrices.)

Solution. From Question 30 we know that

$$p(\lambda) = \lambda^2 - \operatorname{tr}(A)\lambda + \det(A).$$

Thus

$$c_1 = -\operatorname{tr}(A), \quad c_2 = \det(A)$$

Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then

$$\begin{aligned} A^2 &= \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2+bc & ab+bd \\ ca+dc & cb+d^2 \end{pmatrix} \\ \operatorname{tr}(A)A &= (a+d)\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2+ad & ab+db \\ ac+dc & ad+d^2 \end{pmatrix} \\ \det(A)I &= \begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix}. \end{aligned}$$

And

$$\begin{aligned} p(A) &= A^2 - \operatorname{tr}(A)\,A + \det(A)I \\ &= \begin{pmatrix} a^2+bc-a^2-ad+ad-bc & ab+bd-ab-db \\ ca+dc-ac-dc & cb+d^2-ad-d^2+ad-bc \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

Question 34. Prove: If a, b, c, d are integers such that $a+b=c+d$, then

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

has integer eigenvalues.

Solution. Using the result from Question 31, the eigenvalues of A are given by

$$\lambda = \frac{1}{2}\left((a+d) \pm \sqrt{(a-d)^2+4bc}\right).$$

Since $a+b=c+d$, we have $a-d=c-b$ and

$$\begin{aligned} \lambda &= \frac{1}{2}\left((a+d) \pm \sqrt{(c-b)^2+4bc}\right) = \frac{1}{2}\left((a+d) \pm \sqrt{(c+b)^2}\right) \\ &= \frac{(a+d) \pm |c+b|}{2}. \end{aligned}$$

Consequently, the two eigenvalues are given by

$$\begin{aligned} \lambda_1 &= \frac{a+d+c+b}{2} = \frac{2(a+b)}{2} = a+b \\ \lambda_2 &= \frac{a+d-(c+b)}{2} = \frac{c+d-b+d-c-b}{2} = \frac{2(d-b)}{2} = d-b \end{aligned}$$

Since a, b, c, d are integers, λ_1 and λ_2 are also integers.

Question 35. Prove: If λ is an eigenvalue of an invertible matrix A and $\boldsymbol{x}$ is a corresponding eigenvector, then $1/\lambda$ is an eigenvalue of A^{-1} and $\boldsymbol{x}$ is a corresponding eigenvector.

Solution. Since λ is an eigenvalue of A and $\boldsymbol{x}$ is a corresponding eigenvector, we have

$$A\boldsymbol{x} = \lambda\boldsymbol{x}.$$

Since A is invertible, we can multiply both sides of the equation by A^{-1} from the left:

$$A^{-1}(A\boldsymbol{x}) = A^{-1}(\lambda \boldsymbol{x}),$$

which gives

$$I\boldsymbol{x} = \lambda A^{-1}\boldsymbol{x}.$$

Thus,

$$A^{-1}\boldsymbol{x} = \frac{1}{\lambda}\boldsymbol{x}.$$

This shows that $\frac{1}{\lambda}$ is an eigenvalue of A^{-1} with the same eigenvector $\boldsymbol{x}$.

Question 36. Prove: If λ is an eigenvalue of A, $\boldsymbol{x}$ is a corresponding eigenvector, and s is a scalar, then $\lambda - s$ is an eigenvalue of $A - sI$ and $\boldsymbol{x}$ is a corresponding eigenvector.

Solution.

$$(A - sI)\boldsymbol{x} = A\boldsymbol{x} - sI\boldsymbol{x} = A\boldsymbol{x} - s\boldsymbol{x} = \lambda\boldsymbol{x} - s\boldsymbol{x} = (\lambda - s)\boldsymbol{x}$$

Question 37. Prove: If λ is an eigenvalue of A and $\boldsymbol{x}$ is a corresponding eigenvector, then $s\lambda$ is an eigenvalue of sA for every scalar s and $\boldsymbol{x}$ is a corresponding eigenvector.

Solution.

$$(sA)\boldsymbol{x} = s(A\boldsymbol{x}) = s(\lambda\boldsymbol{x}) = (s\lambda)\boldsymbol{x}$$

Question 38. Find the eigenvalues and bases for the eigenspaces of

$$A = \begin{pmatrix} -2 & 2 & 3 \\ -2 & 3 & 2 \\ -4 & 2 & 5 \end{pmatrix}$$

and then use Questions 35 and 36 to find the eigenvalues and bases for the eigenspaces of

1. A^{-1}
2. $A - 3I$
3. $A + 2I$

Solution. The characteristic equation of A is

$$-\lambda^3 + 6\lambda^2 - 11\lambda + 6 = 0$$

A has three eigenvalues: $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 3$. The bases for the eigenspaces are as follows

$$\lambda_1 = 1, \text{ basis vector: } \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\lambda_2 = 2, \text{ basis vector: } \begin{pmatrix} \frac{1}{2} \\ 1 \\ 0 \end{pmatrix}$$

$$\lambda_3 = 3, \text{ basis vector: } \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

1. Eigenvalues and bases for eigenspaces for A^{-1}

$$\lambda_1 = 1, \text{ basis vector: } \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\lambda_2 = \frac{1}{2}, \text{ basis vector: } \begin{pmatrix} \frac{1}{2} \\ 1 \\ 0 \end{pmatrix}$$

$$\lambda_3 = \frac{1}{3}, \text{ basis vector: } \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

2. Eigenvalues and bases for eigenspaces for $A - 3I$

$$\lambda_1 = -2, \text{ basis vector: } \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\lambda_2 = -1, \text{ basis vector: } \begin{pmatrix} \frac{1}{2} \\ 1 \\ 0 \end{pmatrix}$$

$$\lambda_3 = 0, \text{ basis vector: } \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

3. Eigenvalues and bases for eigenspaces for $A + 2I$

$$\lambda_1 = 3, \text{ basis vector: } \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\lambda_2 = 4, \text{ basis vector: } \begin{pmatrix} \frac{1}{2} \\ 1 \\ 0 \end{pmatrix}$$

$$\lambda_3 = 5, \text{ basis vector: } \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Question 39. Show that A and B are not similar matrices.

1. $A = \begin{pmatrix} 1 & 1 \\ 3 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 3 & -2 \end{pmatrix}$

2. $A = \begin{pmatrix} 4 & -1 \\ 2 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 4 & 1 \\ 2 & 4 \end{pmatrix}$

3. $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 2 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

4. $A = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 0 & 2 \\ 3 & 0 & 3 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 2 & 0 \\ 0 & 1 & 1 \end{pmatrix}$

Solution.

1. We can compute that $\det(A) = -1$, $\det(B) = -2$. Since the determinant is invariant under similarity transformations, and given that $\det(A) \neq \det(B)$, it follows that A and B cannot be not similar matrices.
2. $\det(A) = 18$, $\det(B) = 14$
3. $\det(A) = 1$, $\det(B) = 0$
4. $\text{rank}\,(A) = 1$, $\text{rank}\,(B) = 2$. Since rank is invariant under similarity transformations, and given that $\text{rank}\,(A) \neq \text{rank}\,(B)$, it follows that A and B cannot be not similar matrices.

Question 40.

A Procedure for Diagonalizing an $n \times n$ Matrix
Step 1. Determine first whether the matrix is actually diagonalizable by searching for n linearly independent eigenvectors. One way to do this is to find a basis for each eigenspace and count the total number of vectors obtained. If there is a total of n vectors, then the matrix is diagonalizable, and if the total is less than n, then it is not.
Step 2. If you ascertained that the matrix is diagonalizable, then form the matrix

$$P = [\boldsymbol{v}_1 \quad \boldsymbol{v}_2 \quad \cdots \quad \boldsymbol{v}_n]$$

whose column vectors are the n basis vectors you obtained in Step 1.
Step 3. $P^{-1}AP$ will be a diagonal matrix whose successive diagonal entries are the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ that correspond to the successive columns of P.

Following the above procedure to find a matrix P that diagonalizes A, and check your work by computing $P^{-1}AP$.

1. $A = \begin{pmatrix} 1 & 0 \\ 6 & -1 \end{pmatrix}$
2. $A = \begin{pmatrix} -14 & 12 \\ -20 & 17 \end{pmatrix}$
3. $A = \begin{pmatrix} 2 & 0 & -2 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$
4. $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$

Solution.

1. The eigenvalues and bases for eigenspaces for A are:

$$\lambda_1 = 1, \quad \text{basis vector}: \begin{pmatrix} \frac{1}{3} \\ 1 \end{pmatrix}$$

$$\lambda_2 = -1, \quad \text{basis vector}: \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

We can take

$$P = \begin{pmatrix} \frac{1}{3} & 0 \\ 1 & 1 \end{pmatrix}$$

And

$$P^{-1}AP = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

2. The eigenvalues and bases for eigenspaces for A are:

$$\lambda_1 = 1, \quad \text{basis vector}: \begin{pmatrix} \frac{4}{5} \\ 1 \end{pmatrix}$$

$$\lambda_2 = 2, \quad \text{basis vector}: \begin{pmatrix} \frac{3}{4} \\ 1 \end{pmatrix}$$

$$P = \begin{pmatrix} \frac{4}{5} & \frac{3}{4} \\ 1 & 1 \end{pmatrix}, \quad P^{-1}AP = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

3.

$$\lambda_1 = 2, \quad \text{basis vector}: \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\lambda_2 = 3, \quad \text{basis vectors}: \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$$

$$P = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad P^{-1}AP = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

4.

$$\lambda_1 = 0, \quad \text{basis vector}: \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

$$\lambda_2 = 1, \quad \text{basis vector}: \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\lambda_3 = 2, \quad \text{basis vector}: \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$P = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}, \quad P^{-1}AP = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Question 41. Let

1. $A = \begin{pmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 0 & 4 \end{pmatrix}$
2. $A = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 2 \end{pmatrix}$

(a) Find the eigenvalues of A.

(b) For each eigenvalue λ, find the rank of the matrix $\lambda I - A$.

(c) Is A diagonalizable? Justify your conclusion.

Solution.

1. The characteristic equation of A is

$$-\lambda^3 + 11\lambda^2 - 39\lambda + 45 = 0$$

(a) The eigenvalues of A are $\lambda_1 = 3$, $\lambda_2 = 5$

(b)

$$3I - A = \begin{pmatrix} -1 & 0 & -1 \\ -2 & 0 & -2 \\ -1 & 0 & -1 \end{pmatrix}$$

The reduced row echelon form of $3I - A$ is

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

hence $\operatorname{rank}(3I - A) = 1$.

$$5I - A = \begin{pmatrix} 1 & 0 & -1 \\ -2 & 2 & -2 \\ -1 & 0 & 1 \end{pmatrix}$$

The reduced row echelon form of $5I - A$ is

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}$$

hence $\operatorname{rank}(5I - A) = 2$.

(c) A is diagonalizable. A basis for the eigenspace corresponding to $\lambda_1 = 3$ is given by the vectors

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

A basis for the eigenspace corresponding to $\lambda_2 = 5$ is given by the vector

$$\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

Since we have three linearly independent eigenvectors, we construct the matrix

$$P = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 1 \end{pmatrix}.$$

Then, the diagonalization of A is given by

$$P^{-1}AP = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

2. The characteristic equation of A is

$$-\lambda^3 + 7\lambda^2 - 16\lambda + 12 = 0$$

   (a) The eigenvalues of A are $\lambda_1 = 2$, $\lambda_2 = 3$

   (b)

$$2I - A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}$$

   The reduced row echelon form of $2I - A$ is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

   hence $\text{rank}\,(2I - A) = 2$.

$$3I - A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$

   The reduced row echelon form of $3I - A$ is

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

   hence $\text{rank}\,(3I - A) = 2$.

   (c) A is not diagonalizable. A basis for the eigenspace corresponding to $\lambda_1 = 2$ is given by the vectors

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

   A basis for the eigenspace corresponding to $\lambda_2 = 3$ is given by the vector

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

   Since A has only two linearly independent eigenvectors, A is not diagonalizable, which is fewer than the required three for a 3×3 matrix, it is not diagonalizable.

For Questions 42 – 44 Find an LU-decomposition of the coefficient matrix - try both Doolittle decomposition and Crout decomposition. Then solve the system using the decomposition.

Question 42.

1. $\begin{pmatrix} 2 & 8 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -2 \\ -2 \end{pmatrix}$

2. $\begin{pmatrix} -5 & -10 \\ 6 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -10 \\ 19 \end{pmatrix}$

3. $\begin{pmatrix} 2 & -2 & -2 \\ 0 & -2 & 2 \\ -1 & 5 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -4 \\ -2 \\ 6 \end{pmatrix}$

4. $\begin{pmatrix} -3 & 12 & -6 \\ 1 & -2 & 2 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -33 \\ 7 \\ -1 \end{pmatrix}$

5. $\begin{pmatrix} 2 & 6 & 2 \\ -3 & -8 & 0 \\ 4 & 9 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix}$

6. $\begin{pmatrix} 3 & -6 & -3 \\ 2 & 0 & 6 \\ -4 & 7 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ -22 \\ 3 \end{pmatrix}$

7. $\begin{pmatrix} -1 & -3 & 2 \\ -6 & -19 & 10 \\ 3 & 9 & -5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -9 \\ -59 \\ 28 \end{pmatrix}$

8. $\begin{pmatrix} -4 & 0 & 1 \\ 8 & 2 & -1 \\ -8 & -6 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -5 \\ 15 \\ -46 \end{pmatrix}$

Solution.

1.
   - Doolittle decomposition

$$L = \begin{pmatrix} 1 & 0 \\ -\frac{1}{2} & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 2 & 8 \\ 0 & 3 \end{pmatrix}, \quad \boldsymbol{y} = \begin{pmatrix} -2 \\ -3 \end{pmatrix}, \quad \boldsymbol{x} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

   - Crout decomposition

$$L = \begin{pmatrix} 2 & 0 \\ -1 & 3 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}, \quad \boldsymbol{y} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}, \quad \boldsymbol{x} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

2.
   - Doolittle decomposition

$$L = \begin{pmatrix} 1 & 0 \\ -\frac{6}{5} & 1 \end{pmatrix}, \quad U = \begin{pmatrix} -5 & -10 \\ 0 & -7 \end{pmatrix}, \quad \boldsymbol{y} = \begin{pmatrix} -10 \\ 7 \end{pmatrix}, \quad \boldsymbol{x} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$$

   - Crout decomposition

$$L = \begin{pmatrix} -5 & 0 \\ 6 & -7 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \quad \boldsymbol{y} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \quad \boldsymbol{x} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$$

3.
   - Doolittle decomposition

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{1}{2} & -2 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 2 & -2 & -2 \\ 0 & -2 & 2 \\ 0 & 0 & 5 \end{pmatrix}, \quad \boldsymbol{y} = \begin{pmatrix} -4 \\ -2 \\ 0 \end{pmatrix}, \quad \boldsymbol{x} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}.$$

   - Crout decomposition

$$L = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ -1 & 4 & 5 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}, \quad \boldsymbol{y} = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \quad \boldsymbol{x} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}.$$

4. - Doolittle decomposition

$$L = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{3} & 1 & 0 \\ 0 & \frac{1}{2} & 1 \end{pmatrix}, \quad U = \begin{pmatrix} -3 & 12 & -6 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \boldsymbol{y} = \begin{pmatrix} -33 \\ -4 \\ 1 \end{pmatrix}, \quad \boldsymbol{x} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}.$$

   - Crout decomposition

$$L = \begin{pmatrix} -3 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & -4 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \boldsymbol{y} = \begin{pmatrix} 11 \\ -2 \\ 1 \end{pmatrix}, \quad \boldsymbol{x} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}.$$

5. - Doolittle decomposition

$$L = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{3}{2} & 1 & 0 \\ 2 & -3 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 2 & 6 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 7 \end{pmatrix}, \quad \boldsymbol{y} = \begin{pmatrix} 2 \\ 5 \\ 14 \end{pmatrix}, \quad \boldsymbol{x} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}.$$

   - Crout decomposition

$$L = \begin{pmatrix} 2 & 0 & 0 \\ -3 & 1 & 0 \\ 4 & -3 & 7 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & 3 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}, \quad \boldsymbol{y} = \begin{pmatrix} 1 \\ 5 \\ 2 \end{pmatrix}, \quad \boldsymbol{x} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}.$$

6. - Doolittle decomposition

$$L = \begin{pmatrix} 1 & 0 & 0 \\ \frac{2}{3} & 1 & 0 \\ -\frac{4}{3} & -\frac{1}{4} & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 3 & -6 & -3 \\ 0 & 4 & 8 \\ 0 & 0 & 2 \end{pmatrix}, \quad \boldsymbol{y} = \begin{pmatrix} 3 \\ -24 \\ 1 \end{pmatrix}, \quad \boldsymbol{x} = \begin{pmatrix} -\frac{25}{2} \\ -7 \\ \frac{1}{2} \end{pmatrix}.$$

   - Crout decomposition

$$L = \begin{pmatrix} 3 & 0 & 0 \\ 2 & 4 & 0 \\ -4 & -1 & 2 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & -2 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}, \quad \boldsymbol{y} = \begin{pmatrix} 1 \\ -6 \\ \frac{1}{2} \end{pmatrix}, \quad \boldsymbol{x} = \begin{pmatrix} -\frac{25}{2} \\ -7 \\ \frac{1}{2} \end{pmatrix}.$$

7. - Doolittle decomposition

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 6 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} -1 & -3 & 2 \\ 0 & -1 & -2 \\ 0 & 0 & 1 \end{pmatrix}, \quad \boldsymbol{y} = \begin{pmatrix} -9 \\ -5 \\ 1 \end{pmatrix}, \quad \boldsymbol{x} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}.$$

   - Crout decomposition

$$L = \begin{pmatrix} -1 & 0 & 0 \\ -6 & -1 & 0 \\ 3 & 0 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & 3 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}, \quad \boldsymbol{y} = \begin{pmatrix} 9 \\ 5 \\ 1 \end{pmatrix}, \quad \boldsymbol{x} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}.$$

8. • Doolittle decomposition

$$L = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 2 & -3 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} -4 & 0 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 5 \end{pmatrix}, \quad \boldsymbol{y} = \begin{pmatrix} -5 \\ 5 \\ -21 \end{pmatrix}, \quad \boldsymbol{x} = \begin{pmatrix} \frac{1}{5} \\ \frac{23}{5} \\ -\frac{21}{5} \end{pmatrix}.$$

• Crout decomposition

$$L = \begin{pmatrix} -4 & 0 & 0 \\ 8 & 2 & 0 \\ -8 & -6 & 5 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & 0 & -\frac{1}{4} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix}, \quad \boldsymbol{y} = \begin{pmatrix} \frac{5}{4} \\ \frac{5}{2} \\ -\frac{21}{5} \end{pmatrix}, \quad \boldsymbol{x} = \begin{pmatrix} \frac{1}{5} \\ \frac{23}{5} \\ -\frac{21}{5} \end{pmatrix}.$$

Question 43.

1. $$\begin{pmatrix} -2 & -1 & -1 & -2 \\ 2 & 4 & 3 & 1 \\ 4 & 2 & -2 & -1 \\ -1 & 0 & 4 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 3 \\ 11 \\ -10 \\ 8 \end{pmatrix}$$

2. $$\begin{pmatrix} 2 & -2 & 0 & 3 \\ -2 & 3 & 2 & -3 \\ 4 & -5 & -4 & 10 \\ -4 & 7 & 10 & -12 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 11 \\ -9 \\ 10 \\ 2 \end{pmatrix}$$

3. $$\begin{pmatrix} 4 & 3 & 4 & -2 \\ 0 & -1 & 3 & 3 \\ 1 & 2 & -3 & 1 \\ 0 & -4 & -2 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -20 \\ 1 \\ 5 \\ 14 \end{pmatrix}$$

4. $$\begin{pmatrix} -5 & 2 & -2 & 2 \\ -10 & 0 & -8 & 7 \\ 20 & -4 & 10 & -7 \\ -20 & 12 & 0 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 16 \\ 25 \\ -49 \\ 52 \end{pmatrix}$$

5. $$\begin{pmatrix} -1 & 1 & 0 & -3 \\ 0 & 4 & -4 & -4 \\ 0 & -2 & 4 & 0 \\ 4 & -3 & -1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 3 \\ -4 \\ -8 \\ -13 \end{pmatrix}$$

6. $$\begin{pmatrix} 1 & -2 & -3 & -3 \\ 3 & -10 & -12 & -5 \\ 5 & -22 & -26 & 0 \\ 2 & 12 & 0 & -18 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -19 \\ -89 \\ -199 \\ 66 \end{pmatrix}$$

7. $$\begin{pmatrix} 1 & 0 & -1 & -4 \\ -2 & 1 & -2 & -2 \\ 4 & -3 & 4 & -2 \\ -2 & -4 & -3 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 11 \\ 4 \\ 2 \\ 24 \end{pmatrix}$$

8. $$\begin{pmatrix} 1 & 1 & -5 & -1 \\ 2 & -1 & -5 & 3 \\ -4 & -16 & 38 & 23 \\ -5 & -20 & 52 & 35 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -19 \\ -25 \\ 122 \\ 158 \end{pmatrix}$$

Solution.

1. • Doolittle decomposition

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ \frac{1}{2} & \frac{1}{6} & -\frac{25}{24} & 1 \end{pmatrix}, \quad U = \begin{pmatrix} -2 & -1 & -1 & -2 \\ 0 & 3 & 2 & -1 \\ 0 & 0 & -4 & -5 \\ 0 & 0 & 0 & -\frac{97}{24} \end{pmatrix}, \quad \boldsymbol{y} = \begin{pmatrix} 3 \\ 14 \\ -4 \\ 0 \end{pmatrix}, \quad \boldsymbol{x} = \begin{pmatrix} -4 \\ 4 \\ 1 \\ 0 \end{pmatrix}.$$

- Crout decomposition

$$L = \begin{pmatrix} -2 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 4 & 0 & -4 & 0 \\ -1 & \frac{1}{2} & \frac{25}{6} & -\frac{97}{24} \end{pmatrix}, \quad U = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} & 1 \\ 0 & 1 & \frac{2}{3} & -\frac{1}{3} \\ 0 & 0 & 1 & \frac{5}{4} \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \boldsymbol{y} = \begin{pmatrix} -\frac{3}{2} \\ \frac{14}{3} \\ 1 \\ 0 \end{pmatrix}, \quad \boldsymbol{x} = \begin{pmatrix} -4 \\ 4 \\ 1 \\ 0 \end{pmatrix}.$$

2. - Doolittle decomposition

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & -1 & 1 & 0 \\ -2 & 3 & -2 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 2 & -2 & 0 & 3 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & -2 & 4 \\ 0 & 0 & 0 & 2 \end{pmatrix}, \quad \boldsymbol{y} = \begin{pmatrix} 11 \\ 2 \\ -10 \\ -2 \end{pmatrix}, \quad \boldsymbol{x} = \begin{pmatrix} 3 \\ -4 \\ 3 \\ -1 \end{pmatrix}.$$

- Crout decomposition

$$L = \begin{pmatrix} 2 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 4 & -1 & -2 & 0 \\ -4 & 3 & 4 & 2 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & -1 & 0 & \frac{3}{2} \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \boldsymbol{y} = \begin{pmatrix} \frac{11}{2} \\ 2 \\ 5 \\ -1 \end{pmatrix}, \quad \boldsymbol{x} = \begin{pmatrix} 3 \\ -4 \\ 3 \\ -1 \end{pmatrix}.$$

3. - Doolittle decomposition

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{1}{4} & -\frac{5}{4} & 1 & 0 \\ 0 & 4 & 56 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 4 & 3 & 4 & -2 \\ 0 & -1 & 3 & 3 \\ 0 & 0 & -\frac{1}{4} & \frac{21}{4} \\ 0 & 0 & 0 & -310 \end{pmatrix}, \quad \boldsymbol{y} = \begin{pmatrix} -20 \\ 1 \\ \frac{45}{4} \\ -620 \end{pmatrix}, \quad \boldsymbol{x} = \begin{pmatrix} 2 \\ -4 \\ -3 \\ 2 \end{pmatrix}.$$

- Crout decomposition

$$L = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & \frac{5}{4} & -\frac{1}{4} & 0 \\ 0 & -4 & -14 & -310 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & \frac{3}{4} & 1 & -\frac{1}{2} \\ 0 & 1 & -3 & -3 \\ 0 & 0 & 1 & -21 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \boldsymbol{y} = \begin{pmatrix} -5 \\ -1 \\ -45 \\ 2 \end{pmatrix}, \quad \boldsymbol{x} = \begin{pmatrix} 2 \\ -4 \\ -3 \\ 2 \end{pmatrix}.$$

4. - Doolittle decomposition

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -4 & -1 & 1 & 0 \\ 4 & -1 & -2 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} -5 & 2 & -2 & 2 \\ 0 & -4 & -4 & 3 \\ 0 & 0 & -2 & 4 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \boldsymbol{y} = \begin{pmatrix} 16 \\ -7 \\ 8 \\ -3 \end{pmatrix}, \quad \boldsymbol{x} = \begin{pmatrix} -2 \\ 2 \\ 2 \\ 3 \end{pmatrix}.$$

- Crout decomposition

$$L = \begin{pmatrix} -5 & 0 & 0 & 0 \\ -10 & -4 & 0 & 0 \\ 20 & 4 & -2 & 0 \\ -20 & 4 & 4 & -1 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & -\frac{2}{5} & \frac{2}{5} & -\frac{2}{5} \\ 0 & 1 & 1 & -\frac{3}{4} \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \boldsymbol{y} = \begin{pmatrix} -\frac{16}{5} \\ \frac{7}{4} \\ -4 \\ 3 \end{pmatrix}, \quad \boldsymbol{x} = \begin{pmatrix} -2 \\ 2 \\ 2 \\ 3 \end{pmatrix}.$$

5.
- Doolittle decomposition

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -\frac{1}{2} & 1 & 0 \\ -4 & \frac{1}{4} & 0 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} -1 & 1 & 0 & -3 \\ 0 & 4 & -4 & -4 \\ 0 & 0 & 2 & -2 \\ 0 & 0 & 0 & -12 \end{pmatrix}, \quad \boldsymbol{y} = \begin{pmatrix} 3 \\ -4 \\ -10 \\ 0 \end{pmatrix}, \quad \boldsymbol{x} = \begin{pmatrix} -9 \\ -6 \\ -5 \\ 0 \end{pmatrix}.$$

- Crout decomposition

$$L = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & -2 & 2 & 0 \\ 4 & 1 & 0 & -12 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & -1 & 0 & 3 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \boldsymbol{y} = \begin{pmatrix} -3 \\ -1 \\ -5 \\ 0 \end{pmatrix}, \quad \boldsymbol{x} = \begin{pmatrix} -9 \\ -6 \\ -5 \\ 0 \end{pmatrix}.$$

6.
- Doolittle decomposition

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 5 & 3 & 1 & 0 \\ 2 & -4 & 3 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & -2 & -3 & -3 \\ 0 & -4 & -3 & 4 \\ 0 & 0 & -2 & 3 \\ 0 & 0 & 0 & -5 \end{pmatrix}, \quad \boldsymbol{y} = \begin{pmatrix} -19 \\ -32 \\ -8 \\ 0 \end{pmatrix}, \quad \boldsymbol{x} = \begin{pmatrix} 3 \\ 5 \\ 4 \\ 0 \end{pmatrix}.$$

- Crout decomposition

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 3 & -4 & 0 & 0 \\ 5 & -12 & -2 & 0 \\ 2 & 16 & -6 & -5 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & -2 & -3 & -3 \\ 0 & 1 & \frac{3}{4} & -1 \\ 0 & 0 & 1 & -\frac{3}{2} \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \boldsymbol{y} = \begin{pmatrix} -19 \\ 8 \\ 4 \\ 0 \end{pmatrix}, \quad \boldsymbol{x} = \begin{pmatrix} 3 \\ 5 \\ 4 \\ 0 \end{pmatrix}.$$

7.
- Doolittle decomposition

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 4 & -3 & 1 & 0 \\ -2 & -4 & \frac{21}{4} & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & 0 & -1 & -4 \\ 0 & 1 & -4 & -10 \\ 0 & 0 & -4 & -16 \\ 0 & 0 & 0 & 34 \end{pmatrix}, \quad \boldsymbol{y} = \begin{pmatrix} 11 \\ 26 \\ 36 \\ -39 \end{pmatrix}, \quad \boldsymbol{x} = \begin{pmatrix} 2 \\ -\frac{53}{17} \\ -\frac{75}{17} \\ -\frac{39}{34} \end{pmatrix}.$$

- Crout decomposition

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 4 & -3 & -4 & 0 \\ -2 & -4 & -21 & 34 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & 0 & -1 & -4 \\ 0 & 1 & -4 & -10 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \boldsymbol{y} = \begin{pmatrix} 11 \\ 26 \\ -9 \\ -\frac{39}{34} \end{pmatrix}, \quad \boldsymbol{x} = \begin{pmatrix} 2 \\ -\frac{53}{17} \\ -\frac{75}{17} \\ -\frac{39}{34} \end{pmatrix}.$$

8. • Doolittle decomposition

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -4 & 4 & 1 & 0 \\ -5 & 5 & -1 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & 1 & -5 & -1 \\ 0 & -3 & 5 & 5 \\ 0 & 0 & -2 & -1 \\ 0 & 0 & 0 & 4 \end{pmatrix}, \quad \boldsymbol{y} = \begin{pmatrix} -19 \\ 13 \\ -6 \\ -8 \end{pmatrix}, \quad \boldsymbol{x} = \begin{pmatrix} 0 \\ -1 \\ 4 \\ -2 \end{pmatrix}.$$

• Crout decomposition

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & -3 & 0 & 0 \\ -4 & -12 & -2 & 0 \\ -5 & -15 & 2 & 4 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & 1 & -5 & -1 \\ 0 & 1 & -\frac{5}{3} & -\frac{5}{3} \\ 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \boldsymbol{y} = \begin{pmatrix} -19 \\ -\frac{13}{3} \\ 3 \\ -2 \end{pmatrix}, \quad \boldsymbol{x} = \begin{pmatrix} 0 \\ -1 \\ 4 \\ -2 \end{pmatrix}.$$

Question 44.

1.

$$\begin{aligned} -x_1 + 5x_2 - 7x_3 &= 14 \\ -3x_1 + 8x_2 - 4x_3 &= 28 \\ x_1 - 3x_2 + 2x_3 &= -10 \end{aligned}$$

2.

$$\begin{aligned} -12x_1 - 24x_2 - 16x_3 &= 8 \\ -3x_1 - 5x_2 - 2x_3 &= 4 \\ 6x_1 + 9x_2 + x_3 &= -11 \end{aligned}$$

3.

$$\begin{aligned} -4x_1 + 14x_2 - 7x_3 &= 107 \\ -x_1 + 2x_2 - x_3 &= 17 \\ 3x_1 - 9x_2 + 7x_3 &= -83 \end{aligned}$$

4.

$$\begin{aligned} 20x_1 - 11x_2 - 29x_3 &= -92 \\ -4x_1 + 2x_2 + 5x_3 &= 15 \\ -8x_1 + 4x_2 + 14x_3 &= 50 \end{aligned}$$

5.

$$\begin{aligned} 4x_1 - 2x_2 &= 4 \\ 2x_1 + 4x_2 &= 2 \\ x_2 - 5x_3 &= 2 \end{aligned}$$

6.

$$\begin{aligned} -9x_1 + 12x_2 - 9x_3 &= 45 \\ -9x_1 + 3x_2 - 6x_3 &= 24 \\ -3x_1 + 3x_2 - 3x_3 &= 12 \end{aligned}$$

7.

$$\begin{aligned} 3x_1 + 27x_2 - 29x_3 &= 28 \\ -x_1 - 4x_2 + 2x_3 + 5x_4 &= -5 \\ -3x_2 - 3x_3 + 13x_4 &= -1 \\ -5x_1 - 23x_2 + 15x_3 + 21x_4 &= -28 \end{aligned}$$

8.

$$\begin{aligned} 4x_1 - 12x_2 - 29x_3 - 12x_4 &= -149 \\ 3x_2 - 4x_3 + x_4 &= -11 \\ x_1 - 3x_2 - x_3 + x_4 &= -18 \\ 3x_1 + 6x_2 - 28x_3 + 4x_4 &= -122 \end{aligned}$$

9.

$$\begin{aligned} -16x_1 + 15x_2 - 21x_3 - 26x_4 &= -86 \\ 20x_2 - 20x_3 - 10x_4 &= -100 \\ 4x_1 + 2x_3 + 4x_4 &= 2 \\ 16x_1 + 5x_2 + 3x_3 + 14x_4 &= -16 \end{aligned}$$

Solution.

1. • Doolittle decomposition

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -1 & -\frac{2}{7} & 1 \end{pmatrix}, \quad U = \begin{pmatrix} -1 & 5 & -7 \\ 0 & -7 & 17 \\ 0 & 0 & -\frac{1}{7} \end{pmatrix}, \quad \boldsymbol{y} = \begin{pmatrix} 14 \\ -14 \\ 0 \end{pmatrix}, \quad \boldsymbol{x} = \begin{pmatrix} -4 \\ 2 \\ 0 \end{pmatrix}.$$

• Crout decomposition

$$L = \begin{pmatrix} -1 & 0 & 0 \\ -3 & -7 & 0 \\ 1 & 2 & -\frac{1}{7} \end{pmatrix}, \quad U = \begin{pmatrix} 1 & -5 & 7 \\ 0 & 1 & -\frac{17}{7} \\ 0 & 0 & 1 \end{pmatrix}, \quad \boldsymbol{y} = \begin{pmatrix} -14 \\ 2 \\ 0 \end{pmatrix}, \quad \boldsymbol{x} = \begin{pmatrix} -4 \\ 2 \\ 0 \end{pmatrix}.$$

2. • Doolittle decomposition

$$L = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{4} & 1 & 0 \\ -\frac{1}{2} & -3 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} -12 & -24 & -16 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{pmatrix}, \quad \boldsymbol{y} = \begin{pmatrix} 8 \\ 2 \\ -1 \end{pmatrix}, \quad \boldsymbol{x} = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}.$$

• Crout decomposition

$$L = \begin{pmatrix} -12 & 0 & 0 \\ -3 & 1 & 0 \\ 6 & -3 & -1 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & 2 & \frac{4}{3} \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}, \quad \boldsymbol{y} = \begin{pmatrix} -\frac{2}{3} \\ 2 \\ 1 \end{pmatrix}, \quad \boldsymbol{x} = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}.$$

3. • Doolittle decomposition

$$L = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{4} & 1 & 0 \\ -\frac{3}{4} & -1 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} -4 & 14 & -7 \\ 0 & -\frac{3}{2} & \frac{3}{4} \\ 0 & 0 & \frac{5}{2} \end{pmatrix}, \quad \boldsymbol{y} = \begin{pmatrix} 107 \\ -\frac{39}{4} \\ -\frac{25}{2} \end{pmatrix}, \quad \boldsymbol{x} = \begin{pmatrix} -4 \\ 4 \\ -5 \end{pmatrix}.$$

• Crout decomposition

$$L = \begin{pmatrix} -4 & 0 & 0 \\ -1 & -\frac{3}{2} & 0 \\ 3 & \frac{3}{2} & \frac{5}{2} \end{pmatrix}, \quad U = \begin{pmatrix} 1 & -\frac{7}{2} & \frac{7}{4} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix}, \quad \boldsymbol{y} = \begin{pmatrix} -\frac{107}{4} \\ \frac{13}{2} \\ -5 \end{pmatrix}, \quad \boldsymbol{x} = \begin{pmatrix} -4 \\ 4 \\ -5 \end{pmatrix}.$$

4. • Doolittle decomposition

$$L = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{5} & 1 & 0 \\ -\frac{2}{5} & 2 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 20 & -11 & -29 \\ 0 & -\frac{1}{5} & -\frac{4}{5} \\ 0 & 0 & 4 \end{pmatrix}, \quad \boldsymbol{y} = \begin{pmatrix} -92 \\ -\frac{17}{5} \\ 20 \end{pmatrix}, \quad \boldsymbol{x} = \begin{pmatrix} 1 \\ -3 \\ 5 \end{pmatrix}.$$

• Crout decomposition

$$L = \begin{pmatrix} 20 & 0 & 0 \\ -4 & -\frac{1}{5} & 0 \\ -8 & -\frac{2}{5} & 4 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & -\frac{11}{20} & -\frac{29}{20} \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix}, \quad \boldsymbol{y} = \begin{pmatrix} -\frac{23}{5} \\ 17 \\ 5 \end{pmatrix}, \quad \boldsymbol{x} = \begin{pmatrix} 1 \\ -3 \\ 5 \end{pmatrix}.$$

5. • Doolittle decomposition

$$L = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & \frac{1}{5} & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 4 & -2 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -5 \end{pmatrix}, \quad \boldsymbol{y} = \begin{pmatrix} 4 \\ 0 \\ 2 \end{pmatrix}, \quad \boldsymbol{x} = \begin{pmatrix} 1 \\ 0 \\ -\frac{2}{5} \end{pmatrix}.$$

- Crout decomposition

$$L=\begin{pmatrix}4&0&0\\2&5&0\\0&1&-5\end{pmatrix},\quad U=\begin{pmatrix}1&-\frac{1}{2}&0\\0&1&0\\0&0&1\end{pmatrix},\quad \boldsymbol{y}=\begin{pmatrix}1\\0\\-\frac{2}{5}\end{pmatrix},\quad \boldsymbol{x}=\begin{pmatrix}1\\0\\-\frac{2}{5}\end{pmatrix}.$$

6. - Doolittle decomposition

$$L=\begin{pmatrix}1&0&0\\1&1&0\\\frac{1}{3}&\frac{1}{9}&1\end{pmatrix},\quad U=\begin{pmatrix}-9&12&-9\\0&-9&3\\0&0&-\frac{1}{3}\end{pmatrix},\quad \boldsymbol{y}=\begin{pmatrix}45\\-21\\-\frac{2}{3}\end{pmatrix},\quad \boldsymbol{x}=\begin{pmatrix}-3\\3\\2\end{pmatrix}.$$

- Crout decomposition

$$L=\begin{pmatrix}-9&0&0\\-9&-9&0\\-3&-1&-\frac{1}{3}\end{pmatrix},\quad U=\begin{pmatrix}1&-\frac{4}{3}&1\\0&1&-\frac{1}{3}\\0&0&1\end{pmatrix},\quad \boldsymbol{y}=\begin{pmatrix}-5\\\frac{7}{3}\\2\end{pmatrix},\quad \boldsymbol{x}=\begin{pmatrix}-3\\3\\2\end{pmatrix}.$$

7. - Doolittle decomposition

$$L=\begin{pmatrix}1&0&0&0\\-\frac{1}{3}&1&0&0\\0&-\frac{3}{5}&1&0\\-\frac{5}{3}&\frac{22}{5}&-\frac{1}{19}&1\end{pmatrix},\quad U=\begin{pmatrix}3&27&-29&0\\0&5&-\frac{23}{3}&5\\0&0&-\frac{38}{5}&16\\0&0&0&-\frac{3}{19}\end{pmatrix},\quad \boldsymbol{y}=\begin{pmatrix}28\\\frac{13}{3}\\\frac{8}{5}\\-\frac{6}{19}\end{pmatrix},\quad \boldsymbol{x}=\begin{pmatrix}3\\5\\4\\2\end{pmatrix}.$$

- Crout decomposition

$$L=\begin{pmatrix}3&0&0&0\\-1&5&0&0\\0&-3&-\frac{38}{5}&0\\-5&22&\frac{2}{5}&-\frac{3}{19}\end{pmatrix},\quad U=\begin{pmatrix}1&9&-\frac{29}{3}&0\\0&1&-\frac{23}{15}&1\\0&0&1&-\frac{40}{19}\\0&0&0&1\end{pmatrix},\quad \boldsymbol{y}=\begin{pmatrix}\frac{28}{3}\\\frac{13}{15}\\-\frac{4}{19}\\2\end{pmatrix},\quad \boldsymbol{x}=\begin{pmatrix}3\\5\\4\\2\end{pmatrix}.$$

8. - Doolittle decomposition

$$L=\begin{pmatrix}1&0&0&0\\0&1&0&0\\\frac{1}{4}&0&1&0\\\frac{3}{4}&5&\frac{11}{5}&1\end{pmatrix},\quad U=\begin{pmatrix}4&-12&-29&-12\\0&3&-4&1\\0&0&\frac{25}{4}&4\\0&0&0&-\frac{4}{5}\end{pmatrix},\quad \boldsymbol{y}=\begin{pmatrix}-149\\-11\\\frac{77}{4}\\\frac{12}{5}\end{pmatrix},\quad \boldsymbol{x}=\begin{pmatrix}2\\4\\5\\-3\end{pmatrix}.$$

- Crout decomposition

$$L = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 1 & 0 & \frac{25}{4} & 0 \\ 3 & 15 & \frac{55}{4} & -\frac{4}{5} \end{pmatrix}, \quad U = \begin{pmatrix} 1 & -3 & -\frac{29}{4} & -3 \\ 0 & 1 & -\frac{4}{3} & \frac{1}{3} \\ 0 & 0 & 1 & \frac{16}{25} \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \boldsymbol{y} = \begin{pmatrix} -\frac{149}{4} \\ -\frac{11}{3} \\ \frac{77}{25} \\ -3 \end{pmatrix}, \quad \boldsymbol{x} = \begin{pmatrix} 2 \\ 4 \\ 5 \\ -3 \end{pmatrix}.$$

9. - Doolittle decomposition

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{1}{4} & \frac{3}{16} & 1 & 0 \\ -1 & 1 & 4 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} -16 & 15 & -21 & -26 \\ 0 & 20 & -20 & -10 \\ 0 & 0 & \frac{1}{2} & -\frac{5}{8} \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}, \quad \boldsymbol{y} = \begin{pmatrix} -86 \\ -100 \\ \frac{77}{4} \\ -79 \end{pmatrix}, \quad \boldsymbol{x} = \begin{pmatrix} 243 \\ -243 \\ -159 \\ -158 \end{pmatrix}.$$

- Crout decomposition

$$L = \begin{pmatrix} -16 & 0 & 0 & 0 \\ 0 & 20 & 0 & 0 \\ 4 & \frac{15}{4} & \frac{1}{2} & 0 \\ 16 & 20 & 2 & \frac{1}{2} \end{pmatrix}, \quad U = \begin{pmatrix} 1 & -\frac{15}{16} & \frac{21}{16} & \frac{13}{8} \\ 0 & 1 & -1 & -\frac{1}{2} \\ 0 & 0 & 1 & -\frac{5}{4} \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \boldsymbol{y} = \begin{pmatrix} \frac{43}{8} \\ -5 \\ \frac{77}{2} \\ -158 \end{pmatrix}, \quad \boldsymbol{x} = \begin{pmatrix} 243 \\ -243 \\ -159 \\ -158 \end{pmatrix}.$$

Question 45. Determine A^{-1} using LU decomposition and verify the correctness of the following identity:

$$A = LU \Longrightarrow A^{-1} = (LU)^{-1} = U^{-1}L^{-1}.$$

Try both Doolittle decomposition and Crout decomposition.

1. $\begin{pmatrix} 5 & -2 & -4 \\ 15 & -10 & -7 \\ -10 & -8 & 25 \end{pmatrix}$

2. $\begin{pmatrix} -1 & 2 & 5 \\ -2 & 0 & 12 \\ 1 & -6 & -1 \end{pmatrix}$

3. $\begin{pmatrix} -1 & 5 & -2 \\ -2 & 8 & -8 \\ -1 & 1 & -12 \end{pmatrix}$

4. $\begin{pmatrix} -2 & 1 & 3 \\ 2 & 2 & 2 \\ 0 & -4 & -3 \end{pmatrix}$

5. $\begin{pmatrix} 4 & -4 & 5 \\ 12 & -16 & 14 \\ -16 & 36 & -14 \end{pmatrix}$

6. $\begin{pmatrix} 2 & -5 & 1 \\ -2 & 4 & -2 \\ -2 & 7 & 2 \end{pmatrix}$

7. $\begin{pmatrix} -1 & -4 & 5 \\ 0 & 2 & -4 \\ -5 & -22 & -26 \end{pmatrix}$

8. $\begin{pmatrix} -3 & -5 & -3 \\ -12 & -19 & -10 \\ 0 & 0 & 1 \end{pmatrix}$

9. $\begin{pmatrix} 3 & 3 & 0 & -5 \\ 15 & 16 & 0 & -23 \\ -3 & 2 & 4 & 10 \\ 0 & 2 & -8 & 12 \end{pmatrix}$

10. $\begin{pmatrix} 1 & -5 & 4 & -2 \\ 4 & -17 & 12 & -8 \\ -2 & -2 & 11 & 9 \\ 5 & -37 & 36 & -8 \end{pmatrix}$

11. $\begin{pmatrix} 1 & 1 & -4 & -2 \\ 3 & 2 & -10 & -2 \\ 2 & -1 & -1 & 13 \\ -1 & -4 & 8 & 5 \end{pmatrix}$

12. $\begin{pmatrix} 3 & 2 & -3 & 5 \\ -6 & -9 & 6 & -11 \\ -6 & 6 & 9 & -10 \\ -15 & 10 & 12 & -16 \end{pmatrix}$

Solution.

1. $A^{-1} = \begin{pmatrix} \frac{153}{20} & -\frac{41}{20} & \frac{13}{20} \\ \frac{61}{8} & -\frac{17}{8} & \frac{5}{8} \\ \frac{11}{2} & -\frac{3}{2} & \frac{1}{2} \end{pmatrix}$

   - Doolittle decomposition

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -2 & 3 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 5 & -2 & -4 \\ 0 & -4 & 5 \\ 0 & 0 & 2 \end{pmatrix}$$

$$L^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 11 & -3 & 1 \end{pmatrix}, \quad U^{-1} = \begin{pmatrix} \frac{1}{5} & -\frac{1}{10} & \frac{13}{20} \\ 0 & -\frac{1}{4} & \frac{5}{8} \\ 0 & 0 & \frac{1}{2} \end{pmatrix}, \quad U^{-1}L^{-1} = \begin{pmatrix} \frac{153}{20} & -\frac{41}{20} & \frac{13}{20} \\ \frac{61}{8} & -\frac{17}{8} & \frac{5}{8} \\ \frac{11}{2} & -\frac{3}{2} & \frac{1}{2} \end{pmatrix}$$

   - Crount decomposition

$$L = \begin{pmatrix} 5 & 0 & 0 \\ 15 & -4 & 0 \\ -10 & -12 & 2 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & -\frac{2}{5} & -\frac{4}{5} \\ 0 & 1 & -\frac{5}{4} \\ 0 & 0 & 1 \end{pmatrix}.$$

$$L^{-1} = \begin{pmatrix} \frac{1}{5} & 0 & 0 \\ \frac{3}{4} & -\frac{1}{4} & 0 \\ \frac{11}{2} & -\frac{3}{2} & \frac{1}{2} \end{pmatrix}, \quad U^{-1} = \begin{pmatrix} 1 & \frac{2}{5} & \frac{13}{10} \\ 0 & 1 & \frac{5}{4} \\ 0 & 0 & 1 \end{pmatrix}, \quad U^{-1}L^{-1} = \begin{pmatrix} \frac{153}{20} & -\frac{41}{20} & \frac{13}{20} \\ \frac{61}{8} & -\frac{17}{8} & \frac{5}{8} \\ \frac{11}{2} & -\frac{3}{2} & \frac{1}{2} \end{pmatrix}$$

2. $A^{-1} = \begin{pmatrix} 9 & -\frac{7}{2} & 3 \\ \frac{5}{4} & -\frac{1}{2} & \frac{1}{4} \\ \frac{3}{2} & -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$

- Doolittle decomposition

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} -1 & 2 & 5 \\ 0 & -4 & 2 \\ 0 & 0 & 2 \end{pmatrix}.$$

$$L^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & -1 & 1 \end{pmatrix}, \quad U^{-1} = \begin{pmatrix} -1 & -\frac{1}{2} & 3 \\ 0 & -\frac{1}{4} & \frac{1}{4} \\ 0 & 0 & \frac{1}{2} \end{pmatrix}, \quad U^{-1}L^{-1} = \begin{pmatrix} 9 & -\frac{7}{2} & 3 \\ \frac{5}{4} & -\frac{1}{2} & \frac{1}{4} \\ \frac{3}{2} & -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

- Crount decomposition

$$L = \begin{pmatrix} -1 & 0 & 0 \\ -2 & -4 & 0 \\ 1 & -4 & 2 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & -2 & -5 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix}.$$

$$L^{-1} = \begin{pmatrix} -1 & 0 & 0 \\ \frac{1}{2} & -\frac{1}{4} & 0 \\ \frac{3}{2} & -\frac{1}{2} & \frac{1}{2} \end{pmatrix}, \quad U^{-1} = \begin{pmatrix} 1 & 2 & 6 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix}, \quad U^{-1}L^{-1} = \begin{pmatrix} 9 & -\frac{7}{2} & 3 \\ \frac{5}{4} & -\frac{1}{2} & \frac{1}{4} \\ \frac{3}{2} & -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

3. $A^{-1} = \begin{pmatrix} 22 & -\frac{29}{2} & 6 \\ 4 & -\frac{5}{2} & 1 \\ -\frac{3}{2} & 1 & -\frac{1}{2} \end{pmatrix}$

- Doolittle decomposition

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} -1 & 5 & -2 \\ 0 & -2 & -4 \\ 0 & 0 & -2 \end{pmatrix}.$$

$$L^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & -2 & 1 \end{pmatrix}, \quad U^{-1} = \begin{pmatrix} -1 & -\frac{5}{2} & 6 \\ 0 & -\frac{1}{2} & 1 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix}, \quad U^{-1}L^{-1} = \begin{pmatrix} 22 & -\frac{29}{2} & 6 \\ 4 & -\frac{5}{2} & 1 \\ -\frac{3}{2} & 1 & -\frac{1}{2} \end{pmatrix}$$

- Crount decomposition

$$L = \begin{pmatrix} -1 & 0 & 0 \\ -2 & -2 & 0 \\ -1 & -4 & -2 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & -5 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}.$$

$$L^{-1} = \begin{pmatrix} -1 & 0 & 0 \\ 1 & -\frac{1}{2} & 0 \\ -\frac{3}{2} & 1 & -\frac{1}{2} \end{pmatrix}, \quad U^{-1} = \begin{pmatrix} 1 & 5 & -12 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}, \quad U^{-1}L^{-1} = \begin{pmatrix} 22 & -\frac{29}{2} & 6 \\ 4 & -\frac{5}{2} & 1 \\ -\frac{3}{2} & 1 & -\frac{1}{2} \end{pmatrix}$$

4. $A^{-1} = \begin{pmatrix} -\frac{1}{11} & \frac{9}{22} & \frac{2}{11} \\ -\frac{3}{11} & -\frac{3}{11} & -\frac{5}{11} \\ \frac{4}{11} & \frac{4}{11} & \frac{3}{11} \end{pmatrix}$

- Doolittle decomposition

$$L = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -\frac{4}{3} & 1 \end{pmatrix}, \quad U = \begin{pmatrix} -2 & 1 & 3 \\ 0 & 3 & 5 \\ 0 & 0 & \frac{11}{3} \end{pmatrix}.$$

$$L^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ \frac{4}{3} & \frac{4}{3} & 1 \end{pmatrix}, \quad U^{-1} = \begin{pmatrix} -\frac{1}{2} & \frac{1}{6} & \frac{2}{11} \\ 0 & \frac{1}{3} & -\frac{5}{11} \\ 0 & 0 & \frac{3}{11} \end{pmatrix}, \quad U^{-1}L^{-1} = \begin{pmatrix} -\frac{1}{11} & \frac{9}{22} & \frac{2}{11} \\ -\frac{3}{11} & -\frac{3}{11} & -\frac{5}{11} \\ \frac{4}{11} & \frac{4}{11} & \frac{3}{11} \end{pmatrix}$$

- Crount decomposition

$$L = \begin{pmatrix} -2 & 0 & 0 \\ 2 & 3 & 0 \\ 0 & -4 & \frac{11}{3} \end{pmatrix}, \quad U = \begin{pmatrix} 1 & -\frac{1}{2} & -\frac{3}{2} \\ 0 & 1 & \frac{5}{3} \\ 0 & 0 & 1 \end{pmatrix}.$$

$$L^{-1} = \begin{pmatrix} -\frac{1}{2} & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & 0 \\ \frac{4}{11} & \frac{4}{11} & \frac{3}{11} \end{pmatrix}, \quad U^{-1} = \begin{pmatrix} 1 & \frac{1}{2} & \frac{2}{3} \\ 0 & 1 & -\frac{5}{3} \\ 0 & 0 & 1 \end{pmatrix}, \quad U^{-1}L^{-1} = \begin{pmatrix} -\frac{1}{11} & \frac{9}{22} & \frac{2}{11} \\ -\frac{3}{11} & -\frac{3}{11} & -\frac{5}{11} \\ \frac{4}{11} & \frac{4}{11} & \frac{3}{11} \end{pmatrix}$$

5. $A^{-1} = \begin{pmatrix} \frac{35}{2} & -\frac{31}{4} & -\frac{3}{2} \\ \frac{7}{2} & -\frac{3}{2} & -\frac{1}{4} \\ -11 & 5 & 1 \end{pmatrix}$

- Doolittle decomposition

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -4 & -5 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 4 & -4 & 5 \\ 0 & -4 & -1 \\ 0 & 0 & 1 \end{pmatrix}.$$

$$L^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -11 & 5 & 1 \end{pmatrix}, \quad U^{-1} = \begin{pmatrix} \frac{1}{4} & -\frac{1}{4} & -\frac{3}{2} \\ 0 & -\frac{1}{4} & -\frac{1}{4} \\ 0 & 0 & 1 \end{pmatrix}, \quad U^{-1}L^{-1} = \begin{pmatrix} \frac{35}{2} & -\frac{31}{4} & -\frac{3}{2} \\ \frac{7}{2} & -\frac{3}{2} & -\frac{1}{4} \\ -11 & 5 & 1 \end{pmatrix}$$

- Crount decomposition

$$L = \begin{pmatrix} 4 & 0 & 0 \\ 12 & -4 & 0 \\ -16 & 20 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & -1 & \frac{5}{4} \\ 0 & 1 & \frac{1}{4} \\ 0 & 0 & 1 \end{pmatrix}.$$

$$L^{-1} = \begin{pmatrix} \frac{1}{4} & 0 & 0 \\ \frac{3}{4} & -\frac{1}{4} & 0 \\ -11 & 5 & 1 \end{pmatrix}, \quad U^{-1} = \begin{pmatrix} 1 & 1 & -\frac{3}{2} \\ 0 & 1 & -\frac{1}{4} \\ 0 & 0 & 1 \end{pmatrix}, \quad U^{-1}L^{-1} = \begin{pmatrix} \frac{35}{2} & -\frac{31}{4} & -\frac{3}{2} \\ \frac{7}{2} & -\frac{3}{2} & -\frac{1}{4} \\ -11 & 5 & 1 \end{pmatrix}$$

6. $A^{-1} = \begin{pmatrix} -11 & -\frac{17}{2} & -3 \\ -4 & -3 & -1 \\ 3 & 2 & 1 \end{pmatrix}$

- Doolittle decomposition

$$L = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & -2 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 2 & -5 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & 1 \end{pmatrix}.$$

$$L^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix}, \quad U^{-1} = \begin{pmatrix} \frac{1}{2} & -\frac{5}{2} & -3 \\ 0 & -1 & -1 \\ 0 & 0 & 1 \end{pmatrix}, \quad U^{-1}L^{-1} = \begin{pmatrix} -11 & -\frac{17}{2} & -3 \\ -4 & -3 & -1 \\ 3 & 2 & 1 \end{pmatrix}$$

- Crount decomposition

$$L = \begin{pmatrix} 2 & 0 & 0 \\ -2 & -1 & 0 \\ -2 & 2 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & -\frac{5}{2} & \frac{1}{2} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

$$L^{-1} = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ -1 & -1 & 0 \\ 3 & 2 & 1 \end{pmatrix}, \quad U^{-1} = \begin{pmatrix} 1 & \frac{5}{2} & -3 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}, \quad U^{-1}L^{-1} = \begin{pmatrix} -11 & -\frac{17}{2} & -3 \\ -4 & -3 & -1 \\ 3 & 2 & 1 \end{pmatrix}$$

7. $A^{-1} = \begin{pmatrix} -\frac{14}{11} & -\frac{107}{55} & \frac{3}{55} \\ \frac{2}{11} & \frac{51}{110} & -\frac{2}{55} \\ \frac{1}{11} & -\frac{1}{55} & -\frac{1}{55} \end{pmatrix}$

- Doolittle decomposition

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 5 & -1 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} -1 & -4 & 5 \\ 0 & 2 & -4 \\ 0 & 0 & -55 \end{pmatrix}.$$

$$L^{-1}=\begin{pmatrix}1&0&0\\0&1&0\\-5&1&1\end{pmatrix},\quad U^{-1}=\begin{pmatrix}-1&-2&\frac{3}{55}\\0&\frac{1}{2}&-\frac{2}{55}\\0&0&-\frac{1}{55}\end{pmatrix},\quad U^{-1}L^{-1}=\begin{pmatrix}-\frac{14}{11}&-\frac{107}{55}&\frac{3}{55}\\\frac{2}{11}&\frac{51}{110}&-\frac{2}{55}\\\frac{1}{11}&-\frac{1}{55}&-\frac{1}{55}\end{pmatrix}$$

- Crount decomposition

$$L=\begin{pmatrix}-1&0&0\\0&2&0\\-5&-2&-55\end{pmatrix},\quad U=\begin{pmatrix}1&4&-5\\0&1&-2\\0&0&1\end{pmatrix}.$$

$$L^{-1}=\begin{pmatrix}-1&0&0\\0&\frac{1}{2}&0\\\frac{1}{11}&-\frac{1}{55}&-\frac{1}{55}\end{pmatrix},\quad U^{-1}=\begin{pmatrix}1&-4&-3\\0&1&2\\0&0&1\end{pmatrix},\quad U^{-1}L^{-1}=\begin{pmatrix}-\frac{14}{11}&-\frac{107}{55}&\frac{3}{55}\\\frac{2}{11}&\frac{51}{110}&-\frac{2}{55}\\\frac{1}{11}&-\frac{1}{55}&-\frac{1}{55}\end{pmatrix}$$

8. $A^{-1}=\begin{pmatrix}\frac{19}{3}&-\frac{5}{3}&\frac{7}{3}\\-4&1&-2\\0&0&1\end{pmatrix}$

- Doolittle decomposition

$$L=\begin{pmatrix}1&0&0\\4&1&0\\0&0&1\end{pmatrix},\quad U=\begin{pmatrix}-3&-5&-3\\0&1&2\\0&0&1\end{pmatrix}.$$

$$L^{-1}=\begin{pmatrix}1&0&0\\-4&1&0\\0&0&1\end{pmatrix},\quad U^{-1}=\begin{pmatrix}-\frac{1}{3}&-\frac{5}{3}&\frac{7}{3}\\0&1&-2\\0&0&1\end{pmatrix},\quad U^{-1}L^{-1}=\begin{pmatrix}\frac{19}{3}&-\frac{5}{3}&\frac{7}{3}\\-4&1&-2\\0&0&1\end{pmatrix}$$

- Crount decomposition

$$L=\begin{pmatrix}-3&0&0\\-12&1&0\\0&0&1\end{pmatrix},\quad U=\begin{pmatrix}1&\frac{5}{3}&1\\0&1&2\\0&0&1\end{pmatrix}.$$

$$L^{-1}=\begin{pmatrix}-\frac{1}{3}&0&0\\-4&1&0\\0&0&1\end{pmatrix},\quad U^{-1}=\begin{pmatrix}1&-\frac{5}{3}&\frac{7}{3}\\0&1&-2\\0&0&1\end{pmatrix},\quad U^{-1}L^{-1}=\begin{pmatrix}\frac{19}{3}&-\frac{5}{3}&\frac{7}{3}\\-4&1&-2\\0&0&1\end{pmatrix}$$

9. $A^{-1}=\begin{pmatrix}-\frac{325}{3}&21&-\frac{11}{3}&-\frac{11}{6}\\57&-11&2&1\\-\frac{129}{4}&\frac{25}{4}&-1&-\frac{5}{8}\\-31&6&-1&-\frac{1}{2}\end{pmatrix}$

- Doolittle decomposition

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 5 & 1 & 0 & 0 \\ -1 & 5 & 1 & 0 \\ 0 & 2 & -2 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 3 & 3 & 0 & -5 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 4 & -5 \\ 0 & 0 & 0 & -2 \end{pmatrix}.$$

$$L^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -5 & 1 & 0 & 0 \\ 26 & -5 & 1 & 0 \\ 62 & -12 & 2 & 1 \end{pmatrix}, \quad U^{-1} = \begin{pmatrix} \frac{1}{3} & -1 & 0 & -\frac{11}{6} \\ 0 & 1 & 0 & 1 \\ 0 & 0 & \frac{1}{4} & -\frac{5}{8} \\ 0 & 0 & 0 & -\frac{1}{2} \end{pmatrix}, \quad U^{-1}L^{-1} = A^{-1}$$

- Crount decomposition

$$L = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 15 & 1 & 0 & 0 \\ -3 & 5 & 4 & 0 \\ 0 & 2 & -8 & -2 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & 1 & 0 & -\frac{5}{3} \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -\frac{5}{4} \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

$$L^{-1} = \begin{pmatrix} \frac{1}{3} & 0 & 0 & 0 \\ -5 & 1 & 0 & 0 \\ \frac{13}{2} & -\frac{5}{4} & \frac{1}{4} & 0 \\ -31 & 6 & -1 & -\frac{1}{2} \end{pmatrix}, \quad U^{-1} = \begin{pmatrix} 1 & -1 & 0 & \frac{11}{3} \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & \frac{5}{4} \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad U^{-1}L^{-1} = A^{-1}$$

10. $A^{-1} = \begin{pmatrix} \frac{68}{9} & \frac{1}{3} & \frac{8}{9} & -\frac{11}{9} \\ \frac{142}{9} & -\frac{7}{3} & \frac{4}{9} & -\frac{10}{9} \\ \frac{77}{6} & -2 & \frac{1}{3} & -\frac{5}{6} \\ -\frac{21}{2} & 2 & 0 & \frac{1}{2} \end{pmatrix}$

- Doolittle decomposition

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 \\ -2 & -4 & 1 & 0 \\ 5 & -4 & 0 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & -5 & 4 & -2 \\ 0 & 3 & -4 & 0 \\ 0 & 0 & 3 & 5 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

$$L^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -4 & 1 & 0 & 0 \\ -14 & 4 & 1 & 0 \\ -21 & 4 & 0 & 1 \end{pmatrix}, \quad U^{-1} = \begin{pmatrix} 1 & \frac{5}{3} & \frac{8}{9} & -\frac{11}{9} \\ 0 & \frac{1}{3} & \frac{4}{9} & -\frac{10}{9} \\ 0 & 0 & \frac{1}{3} & -\frac{5}{6} \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}, \quad U^{-1}L^{-1} = A^{-1}$$

- Crount decomposition

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 4 & 3 & 0 & 0 \\ -2 & -12 & 3 & 0 \\ 5 & -12 & 0 & 2 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & -5 & 4 & -2 \\ 0 & 1 & -\frac{4}{3} & 0 \\ 0 & 0 & 1 & \frac{5}{3} \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

$$L^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\frac{4}{3} & \frac{1}{3} & 0 & 0 \\ -\frac{14}{3} & \frac{4}{3} & \frac{1}{3} & 0 \\ -\frac{21}{2} & 2 & 0 & \frac{1}{2} \end{pmatrix}, \quad U^{-1} = \begin{pmatrix} 1 & 5 & \frac{8}{3} & -\frac{22}{9} \\ 0 & 1 & \frac{4}{3} & -\frac{20}{9} \\ 0 & 0 & 1 & -\frac{5}{3} \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad U^{-1}L^{-1} == A^{-1}$$

11. $A^{-1} = \begin{pmatrix} -276 & 103 & -22 & -12 \\ -127 & 47 & -10 & -6 \\ -113 & 42 & -9 & -5 \\ 24 & -9 & 2 & 1 \end{pmatrix}$

- Doolittle decomposition

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 2 & 3 & 1 & 0 \\ -1 & 3 & -2 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & 1 & -4 & -2 \\ 0 & -1 & 2 & 4 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

$$L^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 \\ 7 & -3 & 1 & 0 \\ 24 & -9 & 2 & 1 \end{pmatrix}, \quad U^{-1} = \begin{pmatrix} 1 & 1 & 2 & -12 \\ 0 & -1 & 2 & -6 \\ 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad U^{-1}L^{-1} = A^{-1}$$

- Crount decomposition

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 3 & -1 & 0 & 0 \\ 2 & -3 & 1 & 0 \\ -1 & -3 & -2 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & 1 & -4 & -2 \\ 0 & 1 & -2 & -4 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

$$L^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 3 & -1 & 0 & 0 \\ 7 & -3 & 1 & 0 \\ 24 & -9 & 2 & 1 \end{pmatrix}, \quad U^{-1} = \begin{pmatrix} 1 & -1 & 2 & -12 \\ 0 & 1 & 2 & -6 \\ 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad U^{-1}L^{-1} = A^{-1}$$

12. $A^{-1} = \begin{pmatrix} -\frac{26}{9} & -\frac{14}{15} & \frac{2}{45} & -\frac{13}{45} \\ -\frac{5}{3} & -\frac{3}{5} & -\frac{1}{15} & -\frac{1}{15} \\ \frac{56}{9} & 2 & \frac{5}{9} & \frac{2}{9} \\ \frac{19}{3} & 2 & \frac{1}{3} & \frac{1}{3} \end{pmatrix}$

- Doolittle decomposition

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ -2 & -2 & 1 & 0 \\ -5 & -4 & -1 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 3 & 2 & -3 & 5 \\ 0 & -5 & 0 & -1 \\ 0 & 0 & 3 & -2 \\ 0 & 0 & 0 & 3 \end{pmatrix}.$$

$$L^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 6 & 2 & 1 & 0 \\ 19 & 6 & 1 & 1 \end{pmatrix}, \quad U^{-1} = \begin{pmatrix} \frac{1}{3} & \frac{2}{15} & \frac{1}{3} & -\frac{13}{45} \\ 0 & -\frac{1}{5} & 0 & -\frac{1}{15} \\ 0 & 0 & \frac{1}{3} & \frac{2}{9} \\ 0 & 0 & 0 & \frac{1}{3} \end{pmatrix}, \quad U^{-1}L^{-1} == A^{-1}$$

- Crount decomposition

$$L = \begin{pmatrix} 3 & 0 & 0 & 0 \\ -6 & -5 & 0 & 0 \\ -6 & 10 & 3 & 0 \\ -15 & 20 & -3 & 3 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & \frac{2}{3} & -1 & \frac{5}{3} \\ 0 & 1 & 0 & \frac{1}{5} \\ 0 & 0 & 1 & -\frac{2}{3} \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

$$L^{-1} = \begin{pmatrix} \frac{1}{3} & 0 & 0 & 0 \\ -\frac{2}{5} & -\frac{1}{5} & 0 & 0 \\ 2 & \frac{2}{3} & \frac{1}{3} & 0 \\ \frac{19}{3} & 2 & \frac{1}{3} & \frac{1}{3} \end{pmatrix}, \quad U^{-1} = \begin{pmatrix} 1 & -\frac{2}{3} & 1 & -\frac{13}{15} \\ 0 & 1 & 0 & -\frac{1}{5} \\ 0 & 0 & 1 & \frac{2}{3} \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad U^{-1}L^{-1} == A^{-1}$$

Question 46. Determine $\det(A)$ using LU decomposition and verify the correctness of the following identity:

$$A = LU \Longrightarrow \det(A) = \det(L)\det(U).$$

Try both Doolittle decomposition and Crout decomposition.

1. $A = \begin{pmatrix} -1 & 0 & -1 \\ -4 & 1 & -4 \\ -4 & -5 & -4 \end{pmatrix}$
2. $A = \begin{pmatrix} -2 & 4 & 3 \\ -4 & 13 & 6 \\ 8 & -36 & -11 \end{pmatrix}$
3. $A = \begin{pmatrix} 1 & 4 & 1 \\ 3 & 17 & 0 \\ 0 & -20 & 13 \end{pmatrix}$
4. $A = \begin{pmatrix} -3 & -1 & 4 \\ -15 & -4 & 23 \\ 3 & 2 & -6 \end{pmatrix}$

5. $A = \begin{pmatrix} -2 & -2 & 4 \\ 2 & 3 & -9 \\ -2 & -7 & 25 \end{pmatrix}$

6. $A = \begin{pmatrix} -2 & -4 & 0 \\ 0 & -3 & -3 \\ 0 & 9 & 6 \end{pmatrix}$

7. $A = \begin{pmatrix} -2 & 4 & 2 & 0 \\ 6 & -8 & -2 & -1 \\ 8 & -24 & -12 & -3 \\ -8 & 0 & -16 & 13 \end{pmatrix}$

8. $A = \begin{pmatrix} 2 & -2 & 4 & -1 \\ -6 & 11 & -9 & 0 \\ 8 & -33 & 2 & 8 \\ 4 & -4 & 11 & -8 \end{pmatrix}$

9. $A = \begin{pmatrix} 4 & 0 & -4 & 1 \\ 8 & -3 & -3 & 6 \\ -4 & -12 & 23 & 11 \\ -16 & 12 & -2 & -7 \end{pmatrix}$

10. $A = \begin{pmatrix} 3 & -4 & 3 & 5 \\ -15 & 24 & -10 & -30 \\ -12 & 20 & -4 & -20 \\ -6 & 28 & 25 & -23 \end{pmatrix}$

Solution.

1. $\det(A) = 0$

   - Doolittle decomposition

   $$L = \begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 4 & -5 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} -1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \det(L) = 1, \quad \det(U) = 0$$

   - Crount decomposition

   $$L = \begin{pmatrix} -1 & 0 & 0 \\ -4 & 1 & 0 \\ -4 & -5 & 0 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \det(L) = 0, \quad \det(U) = 1$$

   In both cases, we have $\det(A) = \det(L)\det(U)$.

2. $\det(A) = -10$

   - Doolittle decomposition

   $$L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -4 & -4 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} -2 & 4 & 3 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \det(L) = 1, \quad \det(U) = -10$$

   - Crount decomposition

   $$L = \begin{pmatrix} -2 & 0 & 0 \\ -4 & 5 & 0 \\ 8 & -20 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & -2 & -\frac{3}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \det(L) = -10, \quad \det(U) = 1$$

3. $\det(A) = 5$

   - Doolittle decomposition

   $$L = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & -4 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & 4 & 1 \\ 0 & 5 & -3 \\ 0 & 0 & 1 \end{pmatrix}, \quad \det(L) = 1, \quad \det(U) = 5$$

- Crount decomposition

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 5 & 0 \\ 0 & -20 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & 4 & 1 \\ 0 & 1 & -\frac{3}{5} \\ 0 & 0 & 1 \end{pmatrix}, \quad \det(L) = 5, \quad \det(U) = 1$$

4. $\det(A) = 15$

- Doolittle decomposition

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} -3 & -1 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & -5 \end{pmatrix}, \quad \det(L) = 1, \quad \det(U) = 15$$

- Crount decomposition

$$L = \begin{pmatrix} -3 & 0 & 0 \\ -15 & 1 & 0 \\ 3 & 1 & -5 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & \frac{1}{3} & -\frac{4}{3} \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}, \quad \det(L) = 15, \quad \det(U) = 1$$

5. $\det(A) = 8$

- Doolittle decomposition

$$L = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -5 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} -2 & -2 & 4 \\ 0 & 1 & -5 \\ 0 & 0 & -4 \end{pmatrix}, \quad \det(L) = 1, \quad \det(U) = 8$$

- Crount decomposition

$$L = \begin{pmatrix} -2 & 0 & 0 \\ 2 & 1 & 0 \\ -2 & -5 & -4 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & 1 & -2 \\ 0 & 1 & -5 \\ 0 & 0 & 1 \end{pmatrix}, \quad \det(L) = 8, \quad \det(U) = 1$$

6. $\det(A) = -18$

- Doolittle decomposition

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} -2 & -4 & 0 \\ 0 & -3 & -3 \\ 0 & 0 & -3 \end{pmatrix}, \quad \det(L) = 1, \quad \det(U) = -18$$

- Crount decomposition

$$L = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 9 & -3 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad \det(L) = -18, \quad \det(U) = 1$$

7. $\det(A) = 32$

- Doolittle decomposition

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 \\ -4 & -2 & 1 & 0 \\ 4 & -4 & -2 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} -2 & 4 & 2 & 0 \\ 0 & 4 & 4 & -1 \\ 0 & 0 & 4 & -5 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \det(L) = 1, \quad \det(U) = 32$$

- Crount decomposition

$$L = \begin{pmatrix} -2 & 0 & 0 & 0 \\ 6 & 4 & 0 & 0 \\ 8 & -8 & 4 & 0 \\ -8 & -16 & -8 & -1 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & -2 & -1 & 0 \\ 0 & 1 & 1 & -\frac{1}{4} \\ 0 & 0 & 1 & -\frac{5}{4} \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \det(L) = 32, \quad \det(U) = 1$$

8. $\det(A) = 30$

- Doolittle decomposition

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 \\ 4 & -5 & 1 & 0 \\ 2 & 0 & 3 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 2 & -2 & 4 & -1 \\ 0 & 5 & 3 & -3 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 3 \end{pmatrix}, \quad \det(L) = 1, \quad \det(U) = 30$$

- Crount decomposition

$$L = \begin{pmatrix} 2 & 0 & 0 & 0 \\ -6 & 5 & 0 & 0 \\ 8 & -25 & 1 & 0 \\ 4 & 0 & 3 & 3 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & -1 & 2 & -\frac{1}{2} \\ 0 & 1 & \frac{3}{5} & -\frac{3}{5} \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \det(L) = 30, \quad \det(U) = 1$$

9. $\det(A) = 60$

- Doolittle decomposition

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -1 & 4 & 1 & 0 \\ -4 & -4 & -2 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 4 & 0 & -4 & 1 \\ 0 & -3 & 5 & 4 \\ 0 & 0 & -1 & -4 \\ 0 & 0 & 0 & 5 \end{pmatrix}, \quad \det(L) = 1, \quad \det(U) = 60$$

- Crount decomposition

$$L = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 8 & -3 & 0 & 0 \\ -4 & -12 & -1 & 0 \\ -16 & 12 & 2 & 5 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & 0 & -1 & \frac{1}{4} \\ 0 & 1 & -\frac{5}{3} & -\frac{4}{3} \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \det(L) = 60, \quad \det(U) = 1$$

10. $\det(A) = 72$

- Doolittle decomposition

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -5 & 1 & 0 & 0 \\ -4 & 1 & 1 & 0 \\ -2 & 5 & 2 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 3 & -4 & 3 & 5 \\ 0 & 4 & 5 & -5 \\ 0 & 0 & 3 & 5 \\ 0 & 0 & 0 & 2 \end{pmatrix}, \quad \det(L) = 1, \quad \det(U) = 72$$

- Crount decomposition

$$L = \begin{pmatrix} 3 & 0 & 0 & 0 \\ -15 & 4 & 0 & 0 \\ -12 & 4 & 3 & 0 \\ -6 & 20 & 6 & 2 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & -\frac{4}{3} & 1 & \frac{5}{3} \\ 0 & 1 & \frac{5}{4} & -\frac{5}{4} \\ 0 & 0 & 1 & \frac{5}{3} \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \det(L) = 72, \quad \det(U) = 1$$