Tutorial 5

Vector spaces and linear independence

Question 1. Consider $\mathbb{R}^2$ together with addition and scalar multiplication defined as follows: for any $\boldsymbol{u} = \begin{pmatrix} u_1, & u_2 \end{pmatrix}$, $\boldsymbol{v} = \begin{pmatrix} v_1, & v_2 \end{pmatrix} \in \mathbb{R}^2$, and any $\alpha \in \mathbb{R}$

$$\boldsymbol{u} + \boldsymbol{v} = \begin{pmatrix} u_1 + v_1, & u_2 + v_2 \end{pmatrix}, \quad \alpha \otimes \boldsymbol{u} = \begin{pmatrix} 0, & \alpha u_2 \end{pmatrix}$$

1. Compute $\boldsymbol{u} + \boldsymbol{v}$ and $\alpha \otimes \boldsymbol{u}$ for $\boldsymbol{u} = \begin{pmatrix} -1, & 2 \end{pmatrix}$, $\boldsymbol{v} = \begin{pmatrix} 3, & 4 \end{pmatrix}$ and $\alpha = 3$.
2. Prove that $(\mathbb{R}^2, +, \otimes)$ is closed under addition and scalar multiplication.
3. Since vector addition in $(\mathbb{R}^2, +, \otimes)$ coincides with standard vector addition in the usual vector space $(\mathbb{R}^2, +, \cdot)$, certain vector space axioms must hold for $(\mathbb{R}^2, +, \otimes)$ because they are known to hold in $(\mathbb{R}^2, +, \cdot)$. Identify which axioms these are.
4. Show that Axioms 5, 6, 7 of a vector space hold in $(\mathbb{R}^2, +, \otimes)$.
5. Show that Axiom 8 does not hold and hence that $(\mathbb{R}^2, +, \otimes)$ is not a vector space

Solution.

1. $$\begin{pmatrix} -1, & 2 \end{pmatrix} + \begin{pmatrix} 3, & 4 \end{pmatrix} = \begin{pmatrix} 2, & 6 \end{pmatrix}, \quad 3 \otimes \begin{pmatrix} -1, & 2 \end{pmatrix} = \begin{pmatrix} 0, & 6 \end{pmatrix}.$$
2. Take any $\boldsymbol{u} = \begin{pmatrix} u_1, & u_2 \end{pmatrix}, \boldsymbol{v} = \begin{pmatrix} v_1, & v_2 \end{pmatrix} \in \mathbb{R}^2$ and any $\alpha \in \mathbb{R}$, then $u_1, u_2, v_1, v_2 \in \mathbb{R}$, we have $u_1 + v_1, u_2 + v_2, \alpha u_2 \in \mathbb{R}$. Thus
$$\boldsymbol{u} + \boldsymbol{v} = \begin{pmatrix} u_1 + v_1, & u_2 + v_2 \end{pmatrix} \in \mathbb{R}^2, \quad \alpha \otimes \boldsymbol{u} = \begin{pmatrix} 0, & \alpha u_2 \end{pmatrix} \in \mathbb{R}^2,$$
proving that $(\mathbb{R}^2, +, \otimes)$ is closed under addition and scalar multiplication.
3. Axioms 1-4 hold in $(\mathbb{R}^2, +, \otimes)$
4. Take any $\boldsymbol{u} = \begin{pmatrix} u_1, & u_2 \end{pmatrix}, \boldsymbol{v} = \begin{pmatrix} v_1, & v_2 \end{pmatrix} \in \mathbb{R}^2$ and any $\alpha, \beta \in \mathbb{R}$. We have
$$\alpha \otimes (\boldsymbol{u} + \boldsymbol{v}) = \alpha \otimes \begin{pmatrix} u_1 + v_1, & u_2 + v_2 \end{pmatrix} = \begin{pmatrix} 0, & \alpha(u_2 + v_2) \end{pmatrix} = \begin{pmatrix} 0, & \alpha u_2 + \alpha v_2 \end{pmatrix},$$
and
$$\alpha \otimes \boldsymbol{u} + \alpha \otimes \boldsymbol{v} = \begin{pmatrix} 0, & \alpha u_2 \end{pmatrix} + \begin{pmatrix} 0, & \alpha v_2 \end{pmatrix} = \begin{pmatrix} 0, & \alpha u_2 + \alpha v_2 \end{pmatrix},$$
thus Axiom 5 holds. Furthermore,
$$(\alpha+\beta)\otimes\boldsymbol{u} = \begin{pmatrix} 0, & (\alpha + \beta)u_2 \end{pmatrix}, \quad \alpha\otimes\boldsymbol{u}+\beta\otimes\boldsymbol{u} = \begin{pmatrix} 0, & \alpha u_2 \end{pmatrix}+\begin{pmatrix} 0, & \beta u_2 \end{pmatrix} = \begin{pmatrix} 0, & (\alpha + \beta)u_2 \end{pmatrix},$$
showing that Axiom 6 holds. Lastly,
$$\alpha(\beta \otimes \boldsymbol{u}) = \alpha \begin{pmatrix} 0, & \beta u_2 \end{pmatrix} = \begin{pmatrix} 0, & \alpha\beta u_2 \end{pmatrix}, \quad (\alpha\beta) \otimes \boldsymbol{u} = \begin{pmatrix} 0, & \alpha\beta u_2 \end{pmatrix}$$
proves that Axiom 7 holds.
5. By definition, we have
$$1 \otimes \begin{pmatrix} 1, & 3 \end{pmatrix} = \begin{pmatrix} 0, & 3 \end{pmatrix} \neq \begin{pmatrix} 1, & 3 \end{pmatrix},$$
demostrating that Axiom 8 does not hold.

Question 2. Consider $\mathbb{R}^2$ together with addition and scalar multiplication defined as follows: for any $\boldsymbol{u} = \begin{pmatrix} u_1, & u_2 \end{pmatrix}$, $\boldsymbol{v} = \begin{pmatrix} v_1, & v_2 \end{pmatrix} \in \mathbb{R}^2$, and any $\alpha \in \mathbb{R}$

$$\boldsymbol{u} \oplus \boldsymbol{v} = \begin{pmatrix} u_1 + v_1 + 1, & u_2 + v_2 + 1 \end{pmatrix}, \quad \alpha\boldsymbol{u} = \begin{pmatrix} \alpha u_1, & \alpha u_2 \end{pmatrix}$$

1. Compute $\boldsymbol{u} \oplus \boldsymbol{v}$ and $\alpha\boldsymbol{u}$ for $\boldsymbol{u} = \begin{pmatrix} 0, & 4 \end{pmatrix}$, $\boldsymbol{v} = \begin{pmatrix} 1, & -3 \end{pmatrix}$ and $\alpha = 2$.
2. Show that $\begin{pmatrix} 0, & 0 \end{pmatrix}$ does not serve as the additive identity in $(\mathbb{R}^2, \oplus, \cdot)$.
3. Prove that the additive identity in $(\mathbb{R}^2, \oplus, \cdot)$ is given by $\begin{pmatrix} -1, & -1 \end{pmatrix}$
4. Show that Axiom 4 holds by finding the additive inverse of any given $\boldsymbol{u} \in \mathbb{R}^2$
5. Identify two vector space axioms that do not hold in $(\mathbb{R}^2, \oplus, \cdot)$.

Solution.

1. $$\boldsymbol{u} \oplus \boldsymbol{v} = \begin{pmatrix} 0+1+1, & 4-3+1 \end{pmatrix} = \begin{pmatrix} 2, & 2 \end{pmatrix}, \quad \alpha\boldsymbol{u} = \begin{pmatrix} 2 \times 0, & 2 \times 4 \end{pmatrix} = \begin{pmatrix} 0, & 8 \end{pmatrix}$$
2. Let $\boldsymbol{u} = \begin{pmatrix} 1, & 1 \end{pmatrix}$, then
$$\begin{pmatrix} 0, & 0 \end{pmatrix} \oplus \boldsymbol{u} = \begin{pmatrix} 2, & 2 \end{pmatrix} \neq \boldsymbol{u}.$$
3. For any $\boldsymbol{u} = \begin{pmatrix} u_1, & u_2 \end{pmatrix} \in \mathbb{R}^2$,
$$\boldsymbol{u} \oplus \begin{pmatrix} -1, & -1 \end{pmatrix} = \begin{pmatrix} u_1 - 1 + 1, & u_2 - 1 + 1 \end{pmatrix} = \boldsymbol{u}$$
thus $\begin{pmatrix} -1, & -1 \end{pmatrix}$ is the additive identity in $(\mathbb{R}^2, \oplus, \cdot)$.
4. For any $\boldsymbol{u} = \begin{pmatrix} u_1, & u_2 \end{pmatrix} \in \mathbb{R}^2$, let $-\boldsymbol{u} = \begin{pmatrix} -u_1 - 2, & -u_2 - 2 \end{pmatrix}$, then
$$\boldsymbol{u} \oplus (-\boldsymbol{u}) = \begin{pmatrix} u_1 + (-u_1 - 2) + 1, & u_2 + (-u_2 - 2) + 1 \end{pmatrix} = \begin{pmatrix} -1, & -1 \end{pmatrix}.$$
Thus, $-\boldsymbol{u}$ is the additive inverse of $\boldsymbol{u}$ in $(\mathbb{R}^2, \oplus, \cdot)$.
5. Let $\boldsymbol{u} = \begin{pmatrix} 1, & 0 \end{pmatrix}$, $\boldsymbol{v} = \begin{pmatrix} 0, & 1 \end{pmatrix}$, $\alpha = 2$, $\beta = 1$ then
$$\alpha(\boldsymbol{u} \oplus \boldsymbol{v}) = 2\begin{pmatrix} 2, & 2 \end{pmatrix} = \begin{pmatrix} 4, & 4 \end{pmatrix}, \quad \alpha\boldsymbol{u} \oplus \alpha\boldsymbol{v} = \begin{pmatrix} 2, & 0 \end{pmatrix} \oplus \begin{pmatrix} 0, & 2 \end{pmatrix} = \begin{pmatrix} 3, & 3 \end{pmatrix}$$
which shows that Axiom 5 does not hold. Furthermore
$$(\alpha + \beta)\boldsymbol{u} = 3\begin{pmatrix} 1, & 0 \end{pmatrix} = \begin{pmatrix} 3, & 0 \end{pmatrix}, \quad \alpha\boldsymbol{u} \oplus \beta\boldsymbol{u} = \begin{pmatrix} 2, & 0 \end{pmatrix} \oplus \begin{pmatrix} 1, & 0 \end{pmatrix} = \begin{pmatrix} 4, & 1 \end{pmatrix},$$
demonstrating that Aiom 6 does not hold.

Question 3. For each of the following sets equipped with the given operations, determine whether it forms a vector space. For those that are not vector spaces identify the vector space axioms that fail.

1. The set of all real numbers with the standard operations of addition and multiplication.
2. The set of all pairs of real numbers of the form $\begin{pmatrix} x, & 0 \end{pmatrix}$ with the standard vector addition and scalar multiplication in $\mathbb{R}^2$.

3. The set of all pairs of real numbers of the form $\begin{pmatrix} x, & y \end{pmatrix}$ such that $x \geq 0$, with the standard vector addition and scalar multiplication in $\mathbb{R}^2$.

4. The set of all $n-$tuples of real numbers that have the form $\begin{pmatrix} x, & x, & \dots, & x \end{pmatrix}$ with the standard vector addition and scalar multiplication in $\mathbb{R}^n$.

5. The set $\mathbb{R}^3$ with the standard vector addition, but with scalar multiplication defined as

$$\alpha \otimes \begin{pmatrix} u_1, & u_2, & u_3 \end{pmatrix} = \begin{pmatrix} \alpha^2 u_1, & \alpha^2 u_2, & \alpha^2 u_3 \end{pmatrix}.$$

6. The set of all invertible 2×2 matrices, together with the standard matrix addition and scalar multiplication.

7. The set of all diagonal 2×2 matrices of the form

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix},$$

together with the standard matrix addition and scalar multiplication

8. The set of all real-valued functions f defined everywhere on the real line satisfying the condition $f(1) = 0$, together with addition and scalar multiplication defined as follows

$$(f+g)(x) = f(x) + g(x), \quad (\alpha f)(x) = \alpha f(x)$$

9. The subset of $\mathbb{R}^2$ consisting of all pairs of the form $\begin{pmatrix} 1, & y \end{pmatrix}$ with the operations

$$\begin{pmatrix} 1, & y \end{pmatrix} \oplus \begin{pmatrix} 1, & y' \end{pmatrix} = \begin{pmatrix} 1, & y + y' \end{pmatrix}, \quad \alpha \otimes \begin{pmatrix} 1, & y \end{pmatrix} = \begin{pmatrix} 1, & \alpha y \end{pmatrix}$$

10. The set of polynomials of the form $a_0 + a_1 x$ with the operations

$$(a_0 + a_1 x) + (b_0 + b_1 x) = (a_0 + b_0) + (a_1 + b_1)x$$

and

$$\alpha(a_0 + a_1 x) = \alpha a_0 + \alpha a_1 x$$

Solution.

1. It is a vector space.

2. It is a subspace of $\mathbb{R}^2$. Let V denote the subset of $\mathbb{R}^2$ consisting of all pairs of the form $\begin{pmatrix} x, & 0 \end{pmatrix}$. Take any two elements $\begin{pmatrix} x_1, & 0 \end{pmatrix}$, $\begin{pmatrix} x_2, & 0 \end{pmatrix}$ from V and any $\alpha \in \mathbb{R}$

$$\begin{pmatrix} x_1, & 0 \end{pmatrix} + \begin{pmatrix} x_2, & 0 \end{pmatrix} = \begin{pmatrix} x_1 + x_2, & 0 \end{pmatrix} \in V, \quad \alpha \begin{pmatrix} x_1, & 0 \end{pmatrix} = \begin{pmatrix} \alpha x_1, & 0 \end{pmatrix} \in V,$$

showing that V is closed under vector addition and scalar multiplication. Furthermore, $\begin{pmatrix} 0, & 0 \end{pmatrix} \in V$, ensuring that V is nonempty. We can conclude that V is a subspace of $\mathbb{R}^2$.

3. It is not a vector space. Let V denote the subset of $\mathbb{R}^2$ of the form $\begin{pmatrix} x, & y \end{pmatrix}$ with $x \geq 0$. V is not closed under scalar multiplication. For example, take $\begin{pmatrix} 1, 0 \end{pmatrix} \in V$ and scalar $-1 \in \mathbb{R}$

$$(-1)\begin{pmatrix} 1, & 0 \end{pmatrix} = \begin{pmatrix} -1, & 0 \end{pmatrix} \notin V.$$

4. It is a vector space. Let V be the subset of $\mathbb{R}^n$ consisting of $n-$tuples of the form $\begin{pmatrix} x, & x, & \dots, & x \end{pmatrix}$. For any $\begin{pmatrix} x_1, & x_1, & \dots, & x_1 \end{pmatrix}$, $\begin{pmatrix} x_2, & x_2, & \dots, & x_2 \end{pmatrix}$ from V and any scalar $\alpha \in \mathbb{R}$

$$\begin{pmatrix} x_1, & x_1, & \dots, & x_1 \end{pmatrix} + \begin{pmatrix} x_2, & x_2, & \dots, & x_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2, & x_1 + x_2, & \dots, & x_1 x_2 \end{pmatrix} \in V,$$

$$\alpha \begin{pmatrix} x_1, & x_1, & \dots, & x_1 \end{pmatrix} = \begin{pmatrix} \alpha x_1, & \alpha x_1, & \dots, & \alpha x_1 \end{pmatrix} \in V$$

Thus V is closed under vector addition and scalar multiplication, hence it is a subspace of $\mathbb{R}^n$.

5. It is not a vector space. Axiom 6 does not hold: take $\begin{pmatrix} 1, & 0, & 0 \end{pmatrix} \in \mathbb{R}^3$ and $2, 3 \in \mathbb{R}$

$$(2+3) \otimes \begin{pmatrix} 1, & 0, & 0 \end{pmatrix} = \begin{pmatrix} 25, & 0, & 0 \end{pmatrix},$$

$$2 \otimes \begin{pmatrix} 1, & 0, & 0 \end{pmatrix} + 3 \otimes \begin{pmatrix} 1, & 0, & 0 \end{pmatrix} = \begin{pmatrix} 4, & 0, & 0 \end{pmatrix} + \begin{pmatrix} 9, & 0, & 0 \end{pmatrix} = \begin{pmatrix} 13, & 0, & 0 \end{pmatrix} \neq (2+3) \otimes \begin{pmatrix} 1, & 0, & 0 \end{pmatrix}$$

6. It is not a vector space. Let V be the set of all invertible 2×2 matrices. V is not closed under addition: for example, $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ are both invertible matrices, but

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

is not an invertible matrix.

Furthermore, Axiom 3 does not hold because the zero matrix is not an element of V.

7. It is a vector space. Let V be the set of 2×2 diagonal matrices. Take any $\begin{pmatrix} a_1 & 0 \\ 0 & b_1 \end{pmatrix}$, $\begin{pmatrix} a_2 & 0 \\ 0 & b_2 \end{pmatrix}$ from V and any $\alpha \in \mathbb{R}$

$$\begin{pmatrix} a_1 & 0 \\ 0 & b_1 \end{pmatrix} + \begin{pmatrix} a_2 & 0 \\ 0 & b_2 \end{pmatrix} = \begin{pmatrix} a_1 + a_2 & 0 \\ 0 & b_1 + b_2 \end{pmatrix} \in V, \quad \alpha \begin{pmatrix} a_1 & 0 \\ 0 & b_1 \end{pmatrix} = \begin{pmatrix} \alpha a_1 & 0 \\ 0 & \alpha b_1 \end{pmatrix} \in V$$

Thus V is closed under addition and scalar multiplication. The zero matrix belongs to V, ensuring that V is nonempty. We can conclude that V is a subspace of $\mathcal{M}_{2\times 2}$.

8. It is a vector space. Let V denote the subset of consisting of function $f(x)$ satisfying $f(1) = 0$. Take any $f(x), g(x) \in V$ and $\alpha \in \mathbb{R}$

$$(f+g)(1) = f(1) + g(1) = 0 + 0 = 0 \Longrightarrow f + g \in V, \quad (\alpha f)(x) = \alpha f(1) = 0 \Longrightarrow \alpha f \in V$$

shows that V is closed under addition and scalar multiplication. The zero function $f(x) = 0$ for all $x \in \mathbb{R}$ belongs to V. We can conclude that V is a subspace of $\mathcal{F}(\mathbb{R}, \mathbb{R})$.

9. It is a vector space. Let V be the set of all pairs of the form $\begin{pmatrix} 1, & y \end{pmatrix}$. Firstly, it is easy to see that V is closed under addition and scalar multiplication. The additive identity is given by $\begin{pmatrix} 1, & 0 \end{pmatrix}$ and the additive inverse of $\begin{pmatrix} 1, & y \end{pmatrix}$ is $\begin{pmatrix} 1, & -y \end{pmatrix}$. It can be proven that the other axioms also hold.

10. It is a vector space. Let V denote the set of polynomials of the form $a_0 + a_1 x$. It is easy to see that V is closed under addition and scalar multiplication. V is nonempty as it contains the zero polynomial 0. Thus it is a subspace of $\mathbb{R}[x]$.

Question 4. Verify Axioms 1, 2, 5, 6, and 7 for the vector space $\mathcal{M}_{2\times 2}$.

Solution. Take any matrices $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, $\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$ and $\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$ from $\mathcal{M}_{2\times 2}$ and any $\alpha, \beta \in \mathbb{R}$

Axiom 1. Since addition is commutative in $\mathbb{R}$, $a_{ij} + b_{ij} = b_{ij} + a_{ij}$, we have

$$\begin{aligned} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} &= \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{pmatrix} = \begin{pmatrix} b_{11}+a_{11} & b_{12}+a_{12} \\ b_{21}+a_{21} & b_{22}+a_{22} \end{pmatrix} \\ &= \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} + \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}. \end{aligned}$$

Axiom 2. By the associativity of addition in $\mathbb{R}$,

$$(a_{ij} + b_{ij}) + c_{ij} = a_{ij} + (b_{ij} + c_{ij}).$$

We have

$$\left[\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}\right] + \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} = \begin{pmatrix} (a_{11}+b_{11})+c_{11} & (a_{12}+b_{12})+c_{12} \\ (a_{21}+b_{21})+c_{21} & (a_{22}+b_{22})+c_{22} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}+(b_{11}+c_{11}) & a_{12}+(b_{12}+c_{12}) \\ a_{21}+(b_{21}+c_{21}) & a_{22}+(b_{22}+c_{22}) \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} + \left[\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} + \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}\right]$$

Axiom 5. By the distributive law of addition over multiplication for real numbers,

$$\alpha(a_{ij} + b_{ij}) = \alpha a_{ij} + \alpha b_{ij}.$$

We have

$$\alpha\left[\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}\right] = \alpha\begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{pmatrix} = \begin{pmatrix} \alpha(a_{11}+b_{11}) & \alpha(a_{12}+b_{12}) \\ \alpha(a_{21}+b_{21}) & \alpha(a_{22}+b_{22}) \end{pmatrix}$$

$$= \begin{pmatrix} \alpha a_{11}+\alpha b_{11} & \alpha a_{12}+\alpha b_{12} \\ \alpha a_{21}+\alpha b_{21} & \alpha a_{22}+\alpha b_{22} \end{pmatrix} = \begin{pmatrix} \alpha a_{11} & \alpha a_{12} \\ \alpha a_{21} & \alpha a_{22} \end{pmatrix} + \begin{pmatrix} \alpha b_{11} & \alpha b_{12} \\ \alpha\alpha b_{21} & \alpha b_{22} \end{pmatrix}$$

$$= \alpha\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} + \alpha\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

Axiom 6. By the distributive law of addition over multiplication for real numbers,

$$(\alpha + \beta)a_{ij} = \alpha a_{ij} + \beta a_{ij}.$$

We have

$$(\alpha+\beta)\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} (\alpha+\beta)a_{11} & (\alpha+\beta)a_{12} \\ (\alpha+\beta)a_{21} & (\alpha+\beta)a_{22} \end{pmatrix} = \begin{pmatrix} \alpha a_{11}+\beta a_{11} & \alpha a_{12}+\beta a_{12} \\ \alpha a_{21}+\beta a_{21} & \alpha a_{22}+\beta a_{22} \end{pmatrix}$$

$$= \begin{pmatrix} \alpha a_{11} & \alpha a_{12} \\ \alpha a_{21} & \alpha a_{22} \end{pmatrix} + \begin{pmatrix} \beta a_{11} & \beta a_{12} \\ \beta a_{21} & \beta a_{22} \end{pmatrix} = \alpha\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} + \beta\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

Axiom 7. By associativity of multiplication for real numbers

$$(\alpha\beta)a_{ij} = \alpha(\beta a_{ij}).$$

We have

$$\alpha\left[\beta\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\right] = \alpha\begin{pmatrix} \beta a_{11} & \beta a_{12} \\ \beta a_{21} & \beta a_{22} \end{pmatrix} = \begin{pmatrix} \alpha(\beta a_{11}) & \alpha(\beta a_{12}) \\ \alpha(\beta a_{21}) & \alpha(\beta a_{22}) \end{pmatrix} = \begin{pmatrix} (\alpha\beta)a_{11} & (\alpha\beta)a_{12} \\ (\alpha\beta)a_{21} & (\alpha\beta)a_{22} \end{pmatrix}$$

$$= (\alpha\beta)\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

Question 5. Verify Axioms 2, 5, 6, 7, and 8 for the vector space $\mathcal{F}(\mathbb{R}, \mathbb{R})$.

Solution. Take any $f, g, h \in \mathcal{F}(\mathbb{R}, \mathbb{R})$ and any $\alpha, \beta \in \mathbb{R}$.

Axiom 2. By the associativity of addition in $\mathbb{R}$

$$(f(x) + g(x)) + h(x) = f(x) + (g(x) + h(x)).$$

We have

$$\begin{aligned} ((f+g)+h)(x) &= (f+g)(x) + h(x) = (f(x)+g(x)) + h(x) = f(x) + (g(x)+h(x)) \\ &= f(x) + (g+h)(x) = (f+(g+h))(x). \end{aligned}$$

Thus $(f+g)+h = f+(g+h)$.

Axiom 5. By the distributive property of multiplication over addition in $\mathbb{R}$,

$$\alpha(f(x) + g(x)) = \alpha f(x) + \alpha g(x).$$

We have

$$\begin{aligned} (\alpha(f+g))(x) &= \alpha((f+g)(x)) = \alpha(f(x)+g(x)) = \alpha f(x) + \alpha g(x) \\ &= (\alpha f)(x) + (\alpha g)(x) \end{aligned}$$

Axiom 6. By the distributive property of multiplication over addition in $\mathbb{R}$,

$$(\alpha + \beta) f(x) = \alpha f(x) + \beta f(x).$$

We have

$$((\alpha+\beta)f)(x) = (\alpha+\beta)f(x) = \alpha f(x) + \beta f(x) = (\alpha f)(x) + (\beta f)(x)$$

Thus $(\alpha+\beta)f = \alpha f + \beta f$.

Axiom 7. By the associativity of real number multiplication,

$$(\alpha\beta) f(x) = \alpha(\beta f(x)).$$

We have

$$((\alpha\beta)f)(x) = (\alpha\beta)f(x) = \alpha(\beta f(x)) = \alpha((\beta f)(x)).$$

Thus $(\alpha\beta)f = \alpha(\beta f)$.

Axiom 8. $(1f)(x) = 1 \times f(x) = f(x)$ shows that $1f = f$.

Question 6. Show that $\mathbb{R}^2$ with the usual addition and scalar multiplication defined as

$$\alpha \begin{pmatrix} u_1, & u_2 \end{pmatrix} = \begin{pmatrix} \alpha u_1, & 0 \end{pmatrix}$$

satisfy Axioms 1-7.

Solution. Take any $\boldsymbol{u} = \begin{pmatrix} u_1, & u_2 \end{pmatrix}$, $\boldsymbol{v} = \begin{pmatrix} v_1, & v_2 \end{pmatrix}$, $\boldsymbol{w} = \begin{pmatrix} w_1, & w_2 \end{pmatrix}$ from $\mathbb{R}^2$ and any $\alpha, \beta \in \mathbb{R}$

Axiom 1. By the commutativity of addition for real numbers,

$$u_i + v_i = v_i + u_i.$$

$$\boldsymbol{u} + \boldsymbol{v} = \begin{pmatrix} u_1 + v_1, & u_2 + v_2 \end{pmatrix} = \begin{pmatrix} v_1 + u_1, & v_2 + u_2 \end{pmatrix} = \boldsymbol{v} + \boldsymbol{u}$$

Axiom 2. By the associativity of addition in $\mathbb{R}$,

$$(u_i + v_i) + w_i = u_i + (v_i + w_i).$$

$$\begin{aligned}(\boldsymbol{u} + \boldsymbol{v}) + \boldsymbol{w} &= \begin{pmatrix} u_1 + v_1, & u_2 + v_2 \end{pmatrix} + \begin{pmatrix} w_1, & w_2 \end{pmatrix} = \begin{pmatrix} (u_1 + v_1) + w_1, & (u_2 + v_2) + w_2 \end{pmatrix} \\ &= \begin{pmatrix} u_1 + (v_1 + w_1), & u_2 + (v_2 + w_2) \end{pmatrix} \\ &= \begin{pmatrix} u_1, & u_2 \end{pmatrix} + \begin{pmatrix} v_1 + w_1, & v_2 + w_2 \end{pmatrix} = \boldsymbol{u} + (\boldsymbol{v} + \boldsymbol{w})\end{aligned}$$

Axiom 3. The zero vector is $\begin{pmatrix} 0, & 0 \end{pmatrix}$

Axiom 4. The additive inverse of $\boldsymbol{u}$ is $\begin{pmatrix} -u_1, & -u_2 \end{pmatrix}$ since

$$\begin{pmatrix} -u_1, & -u_2 \end{pmatrix} + \boldsymbol{u} = \begin{pmatrix} -u_1 + u_1, & -u_2 + u_2 \end{pmatrix} = \begin{pmatrix} 0, & 0 \end{pmatrix}$$

Axiom 5. By the distributive law of addition over multiplication for real numbers,

$$\alpha(u_i + v_i) = \alpha u_i + \alpha v_i.$$

$$\begin{aligned}\alpha(\boldsymbol{u} + \boldsymbol{v}) &= \alpha \begin{pmatrix} u_1 + v_1, & u_2 + v_2 \end{pmatrix} = \begin{pmatrix} \alpha(u_1 + v_1), & \alpha(u_2 + v_2) \end{pmatrix} = \begin{pmatrix} \alpha u_1 + \alpha v_1, & \alpha u_2 + \alpha v_2 \end{pmatrix} \\ &= \begin{pmatrix} \alpha u_1, & \alpha u_2 \end{pmatrix} + \begin{pmatrix} \alpha v_1, & \alpha v_2 \end{pmatrix} = \alpha \boldsymbol{u} + \alpha \boldsymbol{v}\end{aligned}$$

Axiom 6. By the distributive law of addition over multiplication for real numbers,

$$(\alpha + \beta) u_i = \alpha u_i + \beta u_i.$$

$$\begin{aligned}(\alpha + \beta)\boldsymbol{u} &= \begin{pmatrix} (\alpha + \beta) u_1, & (\alpha + \beta) u_2 \end{pmatrix} = \begin{pmatrix} \alpha u_1 + \beta u_1, & \alpha u_2 + \beta u_2 \end{pmatrix} \\ &= \alpha \begin{pmatrix} u_1, & u_2 \end{pmatrix} + \beta \begin{pmatrix} u_1, & u_2 \end{pmatrix} = \alpha \boldsymbol{u} + \beta \boldsymbol{u}.\end{aligned}$$

Axiom 7. By the associativity of real number multiplication,

$$(\alpha\beta) u_i = \alpha(\beta u_i)$$

$$(\alpha\beta)\boldsymbol{u} = \begin{pmatrix} (\alpha\beta) u_1, & (\alpha\beta) u_2 \end{pmatrix} = \begin{pmatrix} \alpha(\beta u_1), & \alpha(\beta u_2) \end{pmatrix} = \alpha(\beta \boldsymbol{u})$$

Question 7. Consider $\mathbb{R}_{>0}$, the set of positive real numbers. Define addition and scalar multiplication as follows: for any $u, v \in \mathbb{R}_{>0}$ and any $\alpha \in \mathbb{R}$

$$u \oplus v = uv, \quad \alpha \otimes u = u^{\alpha}$$

Verify that Axioms 1 – 5, 7, and 8 hold.

Solution. Take any $u, v, w \in \mathbb{R}_{>0}$ and any $\alpha, \beta \in \mathbb{R}$.

Axiom 1. $u \oplus v = uv = vu = v \oplus u$ follows from the commutativity of multiplication in $\mathbb{R}$.

Axiom 2. By the associativity of addition in $\mathbb{R}$

$$u \oplus (v \oplus w) = u(vw) = (uv)w = (u \oplus v) \oplus w$$

Axiom 3. The additive identity is 1:

$$u \oplus 1 = u1 = u.$$

Axiom 4. The additive inverse of u is $\frac{1}{u}$ since

$$u \oplus \frac{1}{u} = u\frac{1}{u} = 1.$$

It follows from $u > 0$ that $\frac{1}{u} > 0$ and we can conclude $\frac{1}{u} \in \mathbb{R}_{>0}$.

Axiom 5.

$$\alpha \otimes (u \oplus v) = \alpha \otimes (uv) = (uv)^\alpha = u^\alpha v^\alpha = (\alpha \otimes u)(\alpha \otimes v) = (\alpha \otimes u) \oplus (\alpha \otimes v)$$

Axiom 7.

$$(\alpha\beta) \otimes u = u^{\alpha\beta} = (u^\beta)^\alpha = \alpha \otimes (u^\beta) = \alpha \otimes (\beta \otimes u)$$

Axiom 8. $1 \otimes u = u^1 = u$.

Question 8. Show that the set of all points in $\mathbb{R}^2$ lying on a line is a subspace of $(\mathbb{R}^2, +, \cdot)$ iff the line passes through the origin.

Solution.

$\Longrightarrow$ If a line is a subspace of $(\mathbb{R}^2, +, \cdot)$, by definition it contains the zero vector $\begin{pmatrix}0, & 0\end{pmatrix}$, which is the origin.

$\Longleftarrow$ If a line passes through the origin, it can be represented by an equation of the form

$$ax + by = 0,$$

for some $a, b \in \mathbb{R}$. Let

$$W = \left\{ \begin{pmatrix}x, & y\end{pmatrix} \;\middle|\; ax + by = 0 \right\}$$

denote such a line. Take any $\begin{pmatrix}x_1, & y_1\end{pmatrix}$, $\begin{pmatrix}x_2, & y_2\end{pmatrix}$ from W and any $\alpha \in \mathbb{R}$. Then

$$ax_1 + by_1 = 0, \quad ax_2 + by_2 = 0.$$

We have

$$\begin{pmatrix}x_1, & y_1\end{pmatrix} + \begin{pmatrix}x_2, & y_2\end{pmatrix} = \begin{pmatrix}x_1 + x_2, & y_1 + y_2\end{pmatrix}.$$

We have

$$a(x_1 + x_2) + b(y_1 + y_2) = (ax_1 + by_1) + (ax_2 + by_2) = 0 \Longrightarrow \begin{pmatrix}x_1, & y_1\end{pmatrix} + \begin{pmatrix}x_2, & y_2\end{pmatrix} \in W.$$

Thus W is closed under addition. Furthermore, W is also closed under scalar multiplication:

$$\alpha \begin{pmatrix}x_1, & y_1\end{pmatrix} = \begin{pmatrix}\alpha x_1, & \alpha y_1\end{pmatrix}, \quad a(\alpha x_1) + b(\alpha y_1) = \alpha(ax_1 + by_1) = 0 \Longrightarrow \alpha \begin{pmatrix}x_1, & y_1\end{pmatrix} \in W.$$

Question 9. Show that the set of all points in $\mathbb{R}^3$ lying in a plane is a subspace of $(\mathbb{R}^3, +, \cdot)$ iff the plane passes through the origin. *Solution.*

$\Longrightarrow$ If a plane is a subspace of $(\mathbb{R}^3, +, \cdot)$, by definition it contains the zero vector $\begin{pmatrix}0, & 0, & 0\end{pmatrix}$, which is the origin.

$\Longleftarrow$ If a plane passes through the origin, it can be represented by an equation of the form

$$ax + by + cz = 0$$

for some $a, b, c \in \mathbb{R}$. Let

$$W = \left\{ \begin{pmatrix} x, & y, & z \end{pmatrix} \mid ax + by + cz = 0 \right\}$$

denote such a plane. Take any $\begin{pmatrix} x_1, & y_1, & z_1 \end{pmatrix}$, $\begin{pmatrix} x_2, & y_2, & z_2 \end{pmatrix}$ from W and any $\alpha \in \mathbb{R}$. Then

$$ax_1 + by_1 + cz_1 = 0, \quad ax_2 + by_2 + cz_2 = 0.$$

We have

$$\begin{pmatrix} x_1, & y_1, & z_1 \end{pmatrix} + \begin{pmatrix} x_2, & y_2, & z_1 \end{pmatrix} = \begin{pmatrix} x_1 + x_2, & y_1 + y_2, & z_1 + z_2 \end{pmatrix}.$$

We have

$$a(x_1 + x_2) + b(y_1 + y_2) + c(z_1 + z_2) = (ax_1 + by_1 + cz_1) + (ax_2 + by_2 + cz_2) = 0$$

$$\Longrightarrow \begin{pmatrix} x_1, & y_1, & z_1 \end{pmatrix} + \begin{pmatrix} x_2, & y_2, & z_2 \end{pmatrix} \in W.$$

Thus W is closed under addition. Furthermore, W is also closed under scalar multiplication:

$$\alpha \begin{pmatrix} x_1, & y_1, & z_1 \end{pmatrix} = \begin{pmatrix} \alpha x_1, & \alpha y_1, & \alpha z_1 \end{pmatrix}, \quad a(\alpha x_1) + b(\alpha y_1) + c(\alpha z_1) = \alpha(ax_1 + by_1 + cz_1) = 0$$

$$\Longrightarrow \alpha \begin{pmatrix} x_1, & y_1, & z_1 \end{pmatrix} \in W.$$

Question 10. Determine which of the following are subspaces of $\mathbb{R}^3$.

1. All vectors of the form $\begin{pmatrix} a, & 0, & 0 \end{pmatrix}$
2. All vectors of the form $\begin{pmatrix} a, & 1, & 1 \end{pmatrix}$
3. All vectors of the form $\begin{pmatrix} a, & b, & c \end{pmatrix}$, where $b = a + c$
4. All vectors of the form $\begin{pmatrix} a, & b, & c \end{pmatrix}$, where $b = a + c + 1$
5. All vectors of the form $\begin{pmatrix} a, & b, & 0 \end{pmatrix}$

Solution.

1. Let W be the set of all vectors of the form $\begin{pmatrix} a, & 0, & 0 \end{pmatrix}$. Take any $\begin{pmatrix} a_1, & 0, & 0 \end{pmatrix}$, $\begin{pmatrix} a_2, & 0, & 0 \end{pmatrix}$ from W and any $\alpha \in \mathbb{R}$

   $$\begin{pmatrix} a_1, & 0, & 0 \end{pmatrix} + \begin{pmatrix} a_2, & 0, & 0 \end{pmatrix} = \begin{pmatrix} a_1 + a_2, & 0, & 0 \end{pmatrix} \in W,$$

   shows that W is closed under addition.

   $$\alpha \begin{pmatrix} a_1, & 0, & 0 \end{pmatrix} = \begin{pmatrix} \alpha a_1, & 0, & 0 \end{pmatrix}$$

   shows that W is closed under scalar multiplication.

   W is nonempty as it contains the zero vector $\begin{pmatrix} 0, & 0, & 0 \end{pmatrix}$. Thus, W is a subspace of $\mathbb{R}^3$.

2. Let W be be the set of all vectors of the form $\begin{pmatrix} a, & 1, & 1 \end{pmatrix}$. Take $\begin{pmatrix} 0, & 1, & 1 \end{pmatrix}$, $\begin{pmatrix} 1, & 1, & 1 \end{pmatrix}$ from W.

   $$\begin{pmatrix} 0, & 1, & 1 \end{pmatrix} + \begin{pmatrix} 1, & 1, & 1 \end{pmatrix} = \begin{pmatrix} 1, & 2, & 2 \end{pmatrix} \notin W$$

   hence W is not a subspace of $\mathbb{R}^3$.

3. Let W be be the set of all vectors of the form $\begin{pmatrix} a, & b, & c \end{pmatrix}$, where $b = a + c$. Take any $\boldsymbol{u} = \begin{pmatrix} a_1, & b_1, & c_1 \end{pmatrix}$ and $\boldsymbol{v} = \begin{pmatrix} a_2, & b_2, & c_2 \end{pmatrix}$ from W and any $\alpha \in \mathbb{R}$, then

$$b_1 = a_1 + c_1, \quad b_2 = a_2 + c_2.$$

   W is closed under addition:

$$\boldsymbol{u} + \boldsymbol{v} = \begin{pmatrix} a_1, & b_1, & c_1 \end{pmatrix} = \begin{pmatrix} a_2, & b_2, & c_2 \end{pmatrix} = \begin{pmatrix} a_1 + a_2, & b_1 + b_2, & c_1 + c_2 \end{pmatrix},$$

$$a_1 + a_2 + c_1 + c_2 = (a_1 + c_1) + (a_2 + c_2) = b_1 + b_2 \Longrightarrow \boldsymbol{u} + \boldsymbol{v} \in W.$$

   W is closed under scalar multiplication:

$$\alpha\boldsymbol{u} = \begin{pmatrix} \alpha a_1, & \alpha b_1, & \alpha c_1 \end{pmatrix}, \quad \alpha a_1 + \alpha c_1 = \alpha(a_1 + c_1) = \alpha b_1 \Longrightarrow \alpha\boldsymbol{u} \in W$$

   W is nonempty as it contains the zero vector $\begin{pmatrix} 0, & 0, & 0 \end{pmatrix}$ Thus W is a subspace of $\mathbb{R}^3$.

4. Let W be the set of all vectors of the form $\begin{pmatrix} a, & b, & c \end{pmatrix}$, where $b = a+c+1$. Take $\begin{pmatrix} 0, & 1, & 0 \end{pmatrix}$ and $\begin{pmatrix} 1, & 2, & 0 \end{pmatrix}$ from W,

$$\begin{pmatrix} 0, & 1, & 0 \end{pmatrix} + \begin{pmatrix} 1, & 2, & 0 \end{pmatrix} = \begin{pmatrix} 1, & 3, & 0 \end{pmatrix} \notin W$$

   hence W is not a subspace of $\mathbb{R}^3$.

5. Let W be the set of all vectors of the form $\begin{pmatrix} a, & b, & 0 \end{pmatrix}$. Take any $\boldsymbol{u} = \begin{pmatrix} a_1, & b_1, & 0 \end{pmatrix}$ and $\boldsymbol{v} = \begin{pmatrix} a_2, & b_2, & 0 \end{pmatrix}$ from W, and any $\alpha \in \mathbb{R}$.

$$\boldsymbol{u} + \boldsymbol{v} = \begin{pmatrix} a_1 + a_2, & b_1 + b_2, & 0 \end{pmatrix} \in W$$

   shows that W is closed under vector addition.

$$\alpha\boldsymbol{u} = \begin{pmatrix} \alpha a_1, & \alpha b_1, & 0 \end{pmatrix} \in W$$

   shows that W is closed under scalar multiplication. The zero vector $\begin{pmatrix} 0, & 0, & 0 \end{pmatrix}$ belongs to W, ensuring that W is nonempty. Thus W is a subspace of $\mathbb{R}^3$.

Question 11. Determine which of the following are subspaces of $\mathcal{M}_{n\times n}$.

1. The set of all diagonal $n \times n$ matrices
2. The set of all $n \times n$ matrices A such that $\det(A) = 0$
3. The set of all $n \times n$ matrices A such that $\operatorname{tr}(A) = 0$
4. The set of all symmetric $n \times n$ matrices
5. The set of all $n \times n$ matrices A such that $A^\top = -A$
6. The set of all $n \times n$ matrices A for which $A\boldsymbol{x} = \boldsymbol{0}$ has only the trivial solution
7. The set of all $n \times n$ matrices A such that $AB = BA$ for some fixed $n \times n$ matrix B.

Solution.

1. Let W denote the set of all diagonal $n \times n$ matrices.

   Consider any matrices $A = (a_{ij})$ and $B = (b_{ij})$ in W, and let $\alpha \in \mathbb{R}$ be a scalar. By the definition of diagonal matrices, we have

$$a_{ij} = 0, \quad b_{ij} = 0, \quad \text{for all } i \neq j.$$

   The (i, j)-entry of the sum $A + B$ is given by

$$(A + B)_{ij} = a_{ij} + b_{ij}.$$

   Since both a_{ij} and b_{ij} are zero for all $i \neq j$, it follows that

$$(A + B)_{ij} = 0 \quad \text{for all } i \neq j.$$

   Thus, $A + B$ is also a diagonal matrix, establishing closure under addition.

   Next, consider the scalar multiple αA. The (i, j)-entry of αA is given by

$$(\alpha A)_{ij} = \alpha a_{ij}.$$

   Since $a_{ij} = 0$ for all $i \neq j$, we obtain

$$(\alpha A)_{ij} = 0 \quad \text{for all } i \neq j.$$

   This confirms that αA remains a diagonal matrix, demonstrating closure under scalar multiplication.

   The zero matrix is diagonal and belongs to W.

   Since W is nonempty and closed under addition and scalar multiplication, it follows that W is a subspace of $\mathcal{M}_n$.

2. Let W be the set of all $n \times n$ matrices A such that $\det(A) = 0$. Take matrix $A = (a_{ij})$ and $B = (b_{ij})$ from W such that

$$a_{ij} = \begin{cases} 1 & i = j = 1 \\ 0 & \text{otherwise} \end{cases}, \quad b_{ij} = \begin{cases} 1 & i = j,\ i \neq 1 \\ 0 & \text{otherwise} \end{cases}$$

   In other words, A and B are of the form

$$A = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}.$$

   And

$$A + B = I_n \notin W,$$

   showing that W is not a subspace of $\mathcal{M}_{n \times n}$.

3. Let W be the set of all $n \times n$ matrices A such that $\operatorname{tr}(A) = 0$. Take any $A = (a_{ij}),\ B = (b_{ij}) \in W$, then

$$\operatorname{tr}(A) = \sum_{i=1}^{n} a_{ii} = 0, \quad \operatorname{tr}(B) = \sum_{i=1}^{n} b_{ii} = 0.$$

   We have

$$\operatorname{tr}(A + B) = \sum_{i=1}^{n} (a_{ii} + b_{ii}) = \operatorname{tr}(A) + \operatorname{tr}(B) = 0 \Longrightarrow A + B \in W.$$

$$\operatorname{tr}(\alpha A) = \sum_{i=1}^{n} (\alpha a_{ii}) = \alpha \sum_{i=1}^{n} a_{ii} = \alpha 0 = 0 \Longrightarrow \alpha A \in W.$$

Thus W is closed under addition and scalar multiplication. Furthermore, the zero matrix belongs to W, ensuring that W is nonempty. We can conclude that W is a subspace of $\mathcal{M}_{n\times n}$.

4. Let W be the set of all symmetric $n \times n$ matrices. Take any $A = (a_{ij})$ and $B = (b_{ij})$ in W. By definition of symmetry, we have

$$a_{ij} = a_{ji}, \quad b_{ij} = b_{ji}, \quad \text{for all } i, j.$$

Now, consider the (i,j)-entry of $A + B$:

$$(A + B)_{ij} = a_{ij} + b_{ij}.$$

Since $a_{ij} = a_{ji}$ and $b_{ij} = b_{ji}$, it follows that

$$(A + B)_{ij} = a_{ij} + b_{ij} = a_{ji} + b_{ji} = (A + B)_{ji}.$$

Thus, $A + B$ is symmetric, which implies $A + B \in W$, proving closure under addition.

Similarly, for any scalar $\alpha \in \mathbb{R}$, the (i,j)-entry of αA is given by

$$(\alpha A)_{ij} = \alpha a_{ij}.$$

Since A is symmetric, we have $a_{ij} = a_{ji}$, so

$$(\alpha A)_{ij} = \alpha a_{ij} = \alpha a_{ji} = (\alpha A)_{ji}.$$

This shows that αA is symmetric, implying $\alpha A \in W$ and proving closure under scalar multiplication.

Furthermore, the zero matrix is symmetric and belongs to W, ensuring that W is nonempty. We can conclude that W is a subspace of $\mathcal{M}_{n\times n}$.

5. Let W be the set of all $n \times n$ matrices A such that $A^\top = -A$. Take any $A, B \in W$, then

$$A^\top = -A, \quad B^\top = -B.$$

We have

$$(A + B)^\top = A^\top + B^\top = -A + (-B) = -(A + B), \quad (\alpha A)^\top = \alpha A^\top = \alpha(-A) = -(\alpha A),$$

thus $A + B \in W$, $\alpha A \in W$. Therefore, W is closed under addition and scalar multiplication.

Furthermore, the zero matrix O satisfies

$$O^\top = -O = O \Longrightarrow O \in W,$$

ensuring that W is nonempty. We can then conclude that W is a subspace of $\mathcal{M}_{n\times n}$.

6. Let W be the set of all $n \times n$ matrices A for which $A\boldsymbol{x} = \mathbf{0}$ has only the trivial solution. Take $I_n, -I_n \in W$. The system

$$(I_n + (-I_n))\boldsymbol{x} = \mathbf{0}$$

is equivalent to

$$O\boldsymbol{x} = \mathbf{0},$$

where O denotes the zero matrix and the system has infinitely many solutions. Thus $I_n + (-I_n) \notin W$, showing that W is not a vector space.

7. Let W be the set of all $n \times n$ matrices A that commute with a fixed $n \times n$ matrix B. To determine whether W is a subspace of $\mathcal{M}_{n\times n}$, we verify closure under addition and scalar multiplication.

   Take any $A_1, A_2 \in W$ and any scalar $\alpha \in \mathbb{R}$. By definition, we have

   $$A_1B = BA_1, \quad A_2B = BA_2.$$

   Adding these two equations, we obtain

   $$(A_1 + A_2)B = A_1B + A_2B = BA_1 + BA_2 = B(A_1 + A_2),$$

   which shows that $A_1 + A_2 \in W$, proving closure under addition.

   Similarly, for scalar multiplication, we compute

   $$(\alpha A_1)B = \alpha(A_1B) = \alpha(BA_1) = B(\alpha A_1),$$

   which implies $\alpha A_1 \in W$, proving closure under scalar multiplication.

   Finally, the zero matrix O satisfies

   $$OB = BO = O \Longrightarrow O \in W,$$

   showing that W is nonempty.

   Thus, we conclude that W is a subspace of $\mathcal{M}_{n\times n}$.

Question 12. Which of the following are subspaces of $\mathbb{R}^\infty$?

1. All sequences $\boldsymbol{v} \in \mathbb{R}^\infty$ of the form $\boldsymbol{v} = \begin{pmatrix} v, & 0, & v, & 0, & v, & 0, & \ldots \end{pmatrix}$.
2. All sequences $\boldsymbol{v} \in \mathbb{R}^\infty$ of the form $\boldsymbol{v} = \begin{pmatrix} v, & 1, & v, & 1, & v, & 1, & \ldots \end{pmatrix}$.
3. All sequences $\boldsymbol{v} \in \mathbb{R}^\infty$ of the form $\boldsymbol{v} = \begin{pmatrix} v, & 2v, & 4v, & 8v, & 16v, & \ldots \end{pmatrix}$.
4. All sequences in $\mathbb{R}^\infty$ whose components are 0 from some point on.

Solution.

1. Let W be the set of all sequences $\boldsymbol{v} \in \mathbb{R}^\infty$ of the form $\boldsymbol{v} = \begin{pmatrix} v, & 0, & v, & 0, & v, & 0, & \ldots \end{pmatrix}$. Take any $\boldsymbol{v}_1 = \begin{pmatrix} v_1, & 0, & v_1, & 0, & v_1, & 0, & \ldots \end{pmatrix}$, $\boldsymbol{v}_2 = \begin{pmatrix} v_2, & 0, & v_2, & 0, & v_2, & 0, & \ldots \end{pmatrix}$ from W and any $\alpha \in \mathbb{R}$. We have

   $$\boldsymbol{v}_1 + \boldsymbol{v}_2 = \begin{pmatrix} v_1 + v_2, & 0, & v_1 + v_2, & 0, & v_1 + v_2, & 0, & \ldots \end{pmatrix} \in W,$$

   $$\alpha v_1 = \begin{pmatrix} \alpha v_1, & 0, & \alpha v_1, & 0, & \alpha v_1, & 0, & \ldots \end{pmatrix} \in W.$$

   Hence W is closed under addition and scalar multiplication.

   Furthermore, the zero vector $\begin{pmatrix} 0, & 0, & \ldots \end{pmatrix} \in W$, showing that W is nonempty.

   We can conclude that W is a subspace of $\mathbb{R}^\infty$.

2. Let W be the set of all sequences $\boldsymbol{v} \in \mathbb{R}^\infty$ of the form $\boldsymbol{v} = \begin{pmatrix} v, & 1, & v, & 1, & v, & 1, & \ldots \end{pmatrix}$. Take $\boldsymbol{v}_1 = \begin{pmatrix} 0, & 1, & 0, & 1, & 0, & 1, & \ldots \end{pmatrix}$, $\boldsymbol{v}_2 = \begin{pmatrix} 1, & 1, & 1, & 1, & 1, & 1, & \ldots \end{pmatrix}$ from W, then

   $$\boldsymbol{v}_1 + \boldsymbol{v}_2 = \begin{pmatrix} 1, & 2, & 1, & 2, & 1, & 2, & \ldots \end{pmatrix}.$$

   Thus $\boldsymbol{v}_1 + \boldsymbol{v}_2 \notin W$. W is not closed under addition and is not a subspace of $\mathbb{R}^\infty$.

3. Let W be the set of all sequences $\boldsymbol{v} \in \mathbb{R}^\infty$ of the form $\boldsymbol{v} = \begin{pmatrix} v, & 2v, & 4v, & 8v, & 16v, & \dots \end{pmatrix}$. Take any $\boldsymbol{v}_1 = \begin{pmatrix} v_1, & 2v_1, & 4v_1, & 8v_1, & 16v_1, & \dots \end{pmatrix}$, $\boldsymbol{v}_2 = \begin{pmatrix} v_2, & 2v_2, & 4v_2, & 8v_2, & 16v_2, & \dots \end{pmatrix}$ from W and any $\alpha \in \mathbb{R}$. We have

$$\begin{aligned} \boldsymbol{v}_1 + \boldsymbol{v}_2 &= \begin{pmatrix} v_1 + v_2, & 2v_1 + 2v_2, & 4v_1 + 4v_2, & 8v_1 + 8v_2, & 16v_1 + 16v_2, & \dots \end{pmatrix} \\ &= \begin{pmatrix} v_1 + v_2, & 2(v_1 + v_2), & 4(v_1 + v_2), & 8(v_1 + v_2), & 16(v_1 + v_2), & \dots \end{pmatrix} \in W \end{aligned}$$

   Furthermore,

$$\begin{aligned} \alpha\boldsymbol{v}_1 &= \begin{pmatrix} \alpha v_1, & \alpha 2v_1, & \alpha 4v_1, & \alpha 8v_1, & \alpha 16v_1, & \dots \end{pmatrix} \\ &= \begin{pmatrix} \alpha v_1, & 2(\alpha v_1), & 4(\alpha v_1), & 8(\alpha v_1), & 16(\alpha v_1), & \dots \end{pmatrix} \in W. \end{aligned}$$

   Therefore, W is closed under addition and scalar multiplication.

   Finally, the zero vector $\begin{pmatrix} 0, & 0, & 0, & \dots \end{pmatrix} \in W$.

   We can conclude that W is a subspace of $\mathbb{R}^\infty$.

4. Let W be the set of all sequences in $\mathbb{R}^\infty$ whose components are 0 from some point on. Take any $\boldsymbol{u} = \begin{pmatrix} u_1, & u_2, & \dots, & u_t, & 0, & 0, & \dots \end{pmatrix}, \boldsymbol{v} \begin{pmatrix} v_1, & v_2, & \dots, & v_s, & 0, & 0, & \dots \end{pmatrix} \in W$, and any $\alpha \in \mathbb{R}$. By definition of W, there exists $t, s \geq 1$ such that

$$u_i = 0, \ \forall i \geq t; \quad v_j = 0, \ \forall j \geq s.$$

   WLOG, assume $s \geq t$. Then the sum of $\boldsymbol{u}$ and $\boldsymbol{v}$ is given by

$$\boldsymbol{u} + \boldsymbol{v} = \begin{pmatrix} v_1 + u_1, & v_2 + u_2, & \dots, & u_t + v_t, & v_{t+1}, & \dots, & v_s, & 0 & 0 & \dots \end{pmatrix} \in W$$

   The scalar multiple of $\boldsymbol{u}$ by α is given by

$$\alpha\boldsymbol{u} = \begin{pmatrix} \alpha u_1, & \alpha u_2, & \dots, & \alpha u_t, & 0, & 0, & \dots \end{pmatrix} \in W.$$

   Therefore, W is closed under addition and scalar multiplication.

   Finally, the zero sequence $\begin{pmatrix} 0, & 0, & \dots \end{pmatrix} \in W$, ensuring that W is nonempty. We can conclude that W is a subspace of $\mathbb{R}^\infty$.

Question 13. Which of the following are linear combinations of $\boldsymbol{u} = \begin{pmatrix} 0, & -2, & 2 \end{pmatrix}$, $\boldsymbol{v} = \begin{pmatrix} 1, & 3, & -1 \end{pmatrix}$

1. $\begin{pmatrix} 2, & 2, & 2 \end{pmatrix}$ 2. $\begin{pmatrix} 0, & 4, & 5 \end{pmatrix}$ 3. $\begin{pmatrix} 0, & 0, & 0 \end{pmatrix}$

Solution.

1. Suppose

$$\begin{pmatrix} 2, & 2, & 2 \end{pmatrix} = \alpha\boldsymbol{u} + \beta\boldsymbol{v} = \begin{pmatrix} \beta, & -2\alpha + 3\beta, & 2\alpha - \beta \end{pmatrix}$$

   for some scalars α and β, which corresponds to the following linear system in the unknowns α and β

$$\begin{aligned} \beta &= 2 \\ -2\alpha + 3\beta &= 2 \\ 2\alpha - \beta &= 2 \end{aligned}$$

   Solving the system gives

$$\alpha = 2, \quad \beta = 2.$$

   Thus $\begin{pmatrix} 2, & 2, & 2 \end{pmatrix} = 2\boldsymbol{u} + 2\boldsymbol{v}$.

2. Suppose
$$\begin{pmatrix}0, & 4, & 5\end{pmatrix} = \alpha\boldsymbol{u} + \beta\boldsymbol{v} = \begin{pmatrix}\beta, & -2\alpha+3\beta, & 2\alpha-\beta\end{pmatrix}.$$
for some scalars α and β, which corresponds to the following linear system in the unknowns α and β
$$\begin{aligned}\beta &= 0\\ -2\alpha+3\beta &= 4\\ 2\alpha-\beta &= 5\end{aligned}$$
The system has no solutions. Thus $\begin{pmatrix}2, & 2, & 2\end{pmatrix}$ is not a linear combination of $\boldsymbol{u}$ and $\boldsymbol{v}$.

3. $\begin{pmatrix}0, & 0, & 0\end{pmatrix} = 0\boldsymbol{u} + 0\boldsymbol{v}$.

Question 14. Express the following as linear combinations of $\boldsymbol{u} = \begin{pmatrix}2, & 1, & 4\end{pmatrix}$, $\boldsymbol{v} = \begin{pmatrix}1, & -2, & 3\end{pmatrix}$ and $\boldsymbol{w} = \begin{pmatrix}3, & 2, & 5\end{pmatrix}$.

1. $\begin{pmatrix}-9, & -7, & -15\end{pmatrix}$ 2. $\begin{pmatrix}6, & 11, & 6\end{pmatrix}$ 3. $\begin{pmatrix}0, & 0, & 0\end{pmatrix}$

Solution.

1. Suppose
$$\begin{pmatrix}-9, & -7, & -15\end{pmatrix} = \alpha_1\boldsymbol{u} + \alpha_2\boldsymbol{v} + \alpha_3\boldsymbol{w}.$$
for some scalars $\alpha_1, \alpha_2, \alpha_3$. Expanding both sides and equating corresponding entries, we obtain the following system of linear equations:
$$\begin{aligned}2\alpha_1+\alpha_2+3\alpha_3 &= -9\\ \alpha_1-2\alpha_2+2\alpha_3 &= -7\\ 4\alpha_1+3\alpha_2+5\alpha_3 &= -15\end{aligned}$$
The coefficient matrix of this system is:
$$\begin{pmatrix}2 & 1 & 3\\ 1 & -2 & 2\\ 4 & 3 & 5\end{pmatrix}$$
Its inverse is
$$\begin{pmatrix}-4 & 1 & 2\\ \frac{3}{4} & -\frac{1}{2} & -\frac{1}{4}\\ \frac{11}{4} & -\frac{1}{2} & -\frac{5}{4}\end{pmatrix}$$
The unique solution of the system is given by
$$\alpha_1 = -1, \quad \alpha_2 = \frac{1}{2}, \quad \alpha_3 = -\frac{5}{2}$$
Thus
$$\begin{pmatrix}-9, & -7, & -15\end{pmatrix} = -\boldsymbol{u} + \frac{1}{2}\boldsymbol{v} - \frac{5}{2}\boldsymbol{w}.$$

2.
$$\begin{pmatrix}6, & 11, & 6\end{pmatrix} = -\boldsymbol{u} - \frac{5}{2}\boldsymbol{v} + \frac{7}{2}\boldsymbol{w}$$

3.
$$\begin{pmatrix}0, & 0, & 0\end{pmatrix} = 0\boldsymbol{u} + 0\boldsymbol{v} + 0\boldsymbol{w}$$

Question 15. Which of the following are linear combinations of

$$A = \begin{pmatrix}4 & 0\\ -2 & -2\end{pmatrix}, \quad B = \begin{pmatrix}1 & -1\\ 2 & 3\end{pmatrix}, \quad C = \begin{pmatrix}0 & 2\\ 1 & 4\end{pmatrix}$$

1. $\begin{pmatrix}6 & -8\\ -1 & -8\end{pmatrix}$
2. $\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$
3. $\begin{pmatrix}-1 & 5\\ 7 & 1\end{pmatrix}$

Solution.

1. Suppose
$$\begin{pmatrix}6 & -8\\ -1 & -8\end{pmatrix} = \alpha_1 A + \alpha_2 B + \alpha_3 C,$$
for some scalars $\alpha_1, \alpha_2, \alpha_3$. Expanding both sides and equating corresponding entries, we obtain the following system of linear equations:
$$\begin{aligned}4\alpha_1 + \alpha_2 &= 6\\ -\alpha_2 + 2\alpha_3 &= -8\\ -2\alpha_1 + 2\alpha_2 + \alpha_3 &= -1\\ -2\alpha_1 + 3\alpha_2 + 4\alpha_3 &= -8\end{aligned}$$
The coefficient matrix associated with the first three equations is given by:
$$\begin{pmatrix}4 & 1 & 0\\ 0 & -1 & 2\\ -2 & 2 & 1\end{pmatrix}$$
Its inverse is
$$\begin{pmatrix}\frac{5}{24} & \frac{1}{24} & -\frac{1}{12}\\ \frac{1}{6} & -\frac{1}{6} & \frac{1}{3}\\ \frac{1}{12} & \frac{5}{12} & \frac{1}{6}\end{pmatrix} \tag{1}$$
Using this inverse, we solve for $\alpha_1, \alpha_2, \alpha_3$, yielding:
$$\alpha_1 = 1, \quad \alpha_2 = 2, \quad \alpha_3 = -3.$$
Substituting these values into the fourth equation, we verify:
$$-2 \times 1 + 3 \times 2 + 4 \times (-3) = -8.$$
Since this holds, we conclude that:
$$\begin{pmatrix}6 & -8\\ -1 & -8\end{pmatrix} = A + 2B - 3C.$$

2.
$$\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix} = 0A + 0B + 0C.$$

3. Suppose
$$\begin{pmatrix} -1 & 5 \\ 7 & 1 \end{pmatrix} = \alpha_1 A + \alpha_2 B + \alpha_3 C,$$
for some scalars $\alpha_1, \alpha_2, \alpha_3$. Expanding both sides and equating corresponding entries, we obtain the following system of linear equations:
$$\begin{aligned} 4\alpha_1 + \alpha_2 &= -1 \\ -\alpha_2 + 2\alpha_3 &= 5 \\ -2\alpha_1 + 2\alpha_2 + \alpha_3 &= 7 \\ -2\alpha_1 + 3\alpha_2 + 4\alpha_3 &= 1 \end{aligned}$$
By Gauss–Jordan elimination, we obtain the reduced row echelon form:
$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right).$$
Since the last row corresponds to the inconsistent equation $0 = 1$, the system has no solutions. This implies that the given matrix $\begin{pmatrix} -1 & 5 \\ 7 & 1 \end{pmatrix}$ is not a linear combination of A, B, C.

Question 16. In each part, determine whether the vectors span $\mathbb{R}^3$.

1. $\boldsymbol{v}_1 = \begin{pmatrix} 2, & 2, & 2 \end{pmatrix}$, $\boldsymbol{v}_2 = \begin{pmatrix} 0, & 0, & 3 \end{pmatrix}$, $\boldsymbol{v}_3 = \begin{pmatrix} 0, & 1, & 1 \end{pmatrix}$
2. $\boldsymbol{v}_1 = \begin{pmatrix} 2, & -1, & 3 \end{pmatrix}$, $\boldsymbol{v}_2 = \begin{pmatrix} 4, & 1, & 2 \end{pmatrix}$, $\boldsymbol{v}_3 = \begin{pmatrix} 8, & -1, & 8 \end{pmatrix}$

Solution.

1. Suppose $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ span $\mathbb{R}^3$. By definition, this means that any vector in $\mathbb{R}^3$ can be expressed as a linear combination of $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$.

   Consider an arbitrary vector $\boldsymbol{u} \in \mathbb{R}^3$, where
$$\boldsymbol{u} = \begin{pmatrix} u_1, & u_2, & u_3 \end{pmatrix}.$$
   By assumption, there exist scalars $\alpha_1, \alpha_2, \alpha_3$ such that
$$\boldsymbol{u} = \alpha_1 \boldsymbol{v}_1 + \alpha_2 \boldsymbol{v}_2 + \alpha_3 \boldsymbol{v}_3.$$
   This equation corresponds to the following system of linear equations:
$$\begin{aligned} 2\alpha_1 &= u_1 \\ 2\alpha_1 + \alpha_3 &= u_2 \\ 2\alpha_1 + 3\alpha_2 + \alpha_3 &= u_3 \end{aligned}$$
   To determine whether this system is consistent for all possible values of u_1, u_2, u_3, we examine the coefficient matrix:
$$A = \begin{pmatrix} 2 & 0 & 0 \\ 2 & 0 & 1 \\ 2 & 3 & 1 \end{pmatrix}.$$

The system is consistent for all $\boldsymbol{u}$ if and only if A is invertible, which holds if and only if $\det(A) \neq 0$.

Computing the determinant:

$$\det(A) = \begin{vmatrix} 2 & 0 & 0 \\ 2 & 0 & 1 \\ 2 & 3 & 1 \end{vmatrix} = -6 \neq 0.$$

Since $\det(A) \neq 0$, the system is consistent for all $\boldsymbol{u} \in \mathbb{R}^3$, confirming that $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ span $\mathbb{R}^3$.

2. Suppose $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ span $\mathbb{R}^3$. By definition, this means that any vector in $\mathbb{R}^3$ can be expressed as a linear combination of $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$.

   Consider an arbitrary vector $\boldsymbol{u} \in \mathbb{R}^3$, where

$$\boldsymbol{u} = \begin{pmatrix} u_1, & u_2, & u_3 \end{pmatrix}.$$

   By assumption, there exist scalars $\alpha_1, \alpha_2, \alpha_3$ such that

$$\boldsymbol{u} = \alpha_1 \boldsymbol{v}_1 + \alpha_2 \boldsymbol{v}_2 + \alpha_3 \boldsymbol{v}_3.$$

   This equation corresponds to the following system of linear equations:

$$\begin{aligned} 2\alpha_1 + 4\alpha_2 + 8\alpha_3 &= u_1 \\ -\alpha_1 + \alpha_2 - \alpha_3 &= u_2 \\ 3\alpha_1 + 2\alpha_2 + 8\alpha_3 &= u_3 \end{aligned}$$

   To determine whether this system is consistent for all possible values of u_1, u_2, u_3, we examine the coefficient matrix:

$$A = \begin{pmatrix} 2 & 4 & 8 \\ -1 & 1 & -1 \\ 3 & 2 & 8 \end{pmatrix}.$$

   The system is consistent for all $\boldsymbol{u}$ if and only if A is invertible, which holds if and only if $\det(A) \neq 0$.

   Computing the determinant:

$$\det(A) = \begin{vmatrix} 2 & 4 & 8 \\ -1 & 1 & -1 \\ 3 & 2 & 8 \end{vmatrix} = 0.$$

   Since $\det(A) = 0$, $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ do not span $\mathbb{R}^3$.

Question 17. Suppose that $\boldsymbol{v}_1 = \begin{pmatrix} 2, & 1, & 0, & 3 \end{pmatrix}$, $\boldsymbol{v}_2 = \begin{pmatrix} 3, & -1, & 5, & 2 \end{pmatrix}$, and $\boldsymbol{v}_3 = \begin{pmatrix} -1, & 0, & 2, & 1 \end{pmatrix}$. Which of the following vectors are in $\operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$?

1. $\begin{pmatrix} 2, & 3, & -7, & 3 \end{pmatrix}$
2. $\begin{pmatrix} 0, & 0, & 0, & 0 \end{pmatrix}$
3. $\begin{pmatrix} 1, & 1, & 1, & 1 \end{pmatrix}$
4. $\begin{pmatrix} -4, & 6, & -13, & 4 \end{pmatrix}$

Solution.

1. Suppose $\begin{pmatrix}2, & 3, & -7, & 3\end{pmatrix}$ is in span$\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$. Then, by definition of span, there exist scalars $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ such that

$$\begin{pmatrix}2, & 3, & -7, & 3\end{pmatrix} = \alpha_1\boldsymbol{v}_1 + \alpha_2\boldsymbol{v}_2 + \alpha_3\boldsymbol{v}_3$$

This equation corresponds to the following system of linear equations:

$$\begin{aligned} 2\alpha_1 + 3\alpha_2 - \alpha_3 &= 2 \\ \alpha_1 - \alpha_2 &= 3 \\ 5\alpha_2 + 2\alpha_3 &= -7 \\ 3\alpha_1 + 2\alpha_2 + \alpha_3 &= 3. \end{aligned}$$

The augmented matrix of the system is given by

$$\left(\begin{array}{ccc|c} 2 & 3 & -1 & 2 \\ 1 & -1 & 0 & 3 \\ 0 & 5 & 2 & -7 \\ 3 & 2 & 1 & 3 \end{array}\right)$$

By Gauss–Jordan elimination, we obtain the reduced row echelon form:

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

From this, we determine the unique solution:

$$\alpha_1 = 2, \quad \alpha_2 = -1, \quad \alpha_3 = -1$$

Thus, we conclude that

$$\begin{pmatrix}2, & 3, & -7, & 3\end{pmatrix} = 2\boldsymbol{v}_1 - \boldsymbol{v}_2 - \boldsymbol{v}_3,$$

which confirms that span$\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$.

2. $\begin{pmatrix}0, & 0, & 0, & 0\end{pmatrix}$ is in span$\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$.

3. $\begin{pmatrix}1, & 1, & 1, & 1\end{pmatrix}$ is not in span$\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$.

4.
$$\begin{pmatrix}-4, & 6, & -13, & 4\end{pmatrix} = 3\boldsymbol{v}_1 - 3\boldsymbol{v}_2 + \boldsymbol{v}_3,$$

is in span$\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$.

Question 18. Determine whether the solution space of the system $A\boldsymbol{x} = \boldsymbol{0}$ is a line through the origin, a plane through the origin, or the origin only. If it is a plane, find an equation for it. If it is a line, find parametric equations for it.

1. $A = \begin{pmatrix} -1 & 1 & 1 \\ 3 & -1 & 0 \\ 2 & -4 & -5 \end{pmatrix}$

2. $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{pmatrix}$

3. $A = \begin{pmatrix} 1 & -3 & 1 \\ 2 & -6 & 2 \\ 3 & -9 & 3 \end{pmatrix}$

4. $A = \begin{pmatrix} 1 & -1 & 1 \\ 2 & -1 & 4 \\ 3 & 1 & 11 \end{pmatrix}$

5. $A = \begin{pmatrix} 10 & 4 & 21 \\ 0 & -4 & 3 \\ -5 & -1 & -12 \end{pmatrix}$

6. $A = \begin{pmatrix} 18 & -9 & -14 \\ 6 & -3 & -5 \\ -3 & 1 & 2 \end{pmatrix}$

7. $A = \begin{pmatrix} 3 & -6 & 9 \\ -2 & 7 & -2 \\ 0 & 1 & 5 \end{pmatrix}$

8. $A = \begin{pmatrix} 3 & 6 & -9 \\ 0 & 0 & -2 \\ -2 & 1 & 5 \end{pmatrix}$

9. $A = \begin{pmatrix} a & b & b \\ b & a & b \\ b & b & a \end{pmatrix}, \quad a \neq 0 \text{ or } b \neq 0$

Solution.

1. The reduced row echelon form of the augmented matrix is:

$$\begin{pmatrix} 1 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & \frac{3}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The solution space is a line through the origin and its parametric equation is

$$x = -\frac{t}{2}, \quad y = -\frac{3t}{2}, \quad z = t, \quad t \in \mathbb{R}.$$

2. The solution space is the zero vector space.

3. The reduced row echelon form of the augmented matrix is:

$$\begin{pmatrix} 1 & -3 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The solution space is a plane:

$$x + z - 3y = 0.$$

4. The reduced row echelon form of the augmented matrix is:

$$\begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The solution space is a line with parametric equation

$$x = -3t, \quad y = -2t, \quad z = t, \quad t \in \mathbb{R}.$$

5. The solution space is the zero vector space.

6. The solution space is the zero vector space.

7. The solution space is the zero vector space.

8. The solution space is the zero vector space.

9. If $a = b \neq 0$, the solution space is a plane with equation

$$x + y + z = 0.$$

If $a \neq b$, we apply Gauss-Jordan elimination to the augmented matrix

$$\left(\begin{array}{ccc|c} a & b & b & 0 \\ b & a & b & 0 \\ b & b & a & 0 \end{array}\right) \xrightarrow[R_1 \to R_1 - R_2]{R_2 \to R_2 - R_3} \left(\begin{array}{ccc|c} a-b & b-a & 0 & 0 \\ 0 & a-b & b-a & 0 \\ b & b & a & 0 \end{array}\right) \xrightarrow[R_2 \to \frac{1}{a-b} R_2]{R_1 \to \frac{1}{a-b} R1} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ b & b & a & 0 \end{array}\right)$$

$$\xrightarrow[R_1 \to R_1 + R_2]{R_3 \to R_3 - bR_2} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ b & 0 & a+b & 0 \end{array}\right) \xrightarrow{R_3 \to R_3 - bR_1} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & a+2b & 0 \end{array}\right)$$

If $a = -2b$, the reduced row echelon form of the augmented matrix is

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

and the solution space is a line with parametric equation

$$x = t, \quad y = t, \quad z = t, \quad t \in \mathbb{R}.$$

If $a \neq -2b$ and $a \neq b$, we continue with Gauss-Jordan elimination

$$\xrightarrow{R_3 \to \frac{1}{a+2b}} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \xrightarrow[R_2 \to R_2 + R_3]{R_1 \to R_1 + R_3} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

and the solution space is the zero vector space.

Question 19. Explain why the following form linearly dependent sets of vectors

1. $\boldsymbol{u}_1 = \begin{pmatrix} -1, & 2, & 4 \end{pmatrix}, \quad \boldsymbol{u}_2 = \begin{pmatrix} 5, & -10, & -20 \end{pmatrix}$ in $\mathbb{R}^3$
2. $\boldsymbol{u}_1 = \begin{pmatrix} 3, & -1 \end{pmatrix}, \quad \boldsymbol{u}_2 = \begin{pmatrix} 4, & 5 \end{pmatrix}, \quad \boldsymbol{u}_3 = \begin{pmatrix} -2, & 7 \end{pmatrix}$ in $\mathbb{R}^2$
3. $A = \begin{pmatrix} -3 & 4 \\ 2 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & -4 \\ -2 & 0 \end{pmatrix}$ in $\mathcal{M}_{2\times 2}$

Solution.

1. $\boldsymbol{u}_2 = -5\boldsymbol{u}_1$. $\boldsymbol{u}_2$ is a linear combination of $\boldsymbol{u}_1$, thus the set $\{\boldsymbol{u}_1, \boldsymbol{u}_2\}$ is linearly dependent.
2. $\boldsymbol{u}_3 = -2\boldsymbol{u}_1 + \boldsymbol{u}_2$
3. $B = -A$

Question 20. In each part, determine whether the vectors are linearly independent or are linearly dependent in $\mathbb{R}^3$.

1. $\begin{pmatrix}-3, & 0, & 4\end{pmatrix}, \quad \begin{pmatrix}5, & -1, & 2\end{pmatrix}, \quad \begin{pmatrix}1, & 1, & 3\end{pmatrix}$
2. $\begin{pmatrix}-2, & 0, & 1\end{pmatrix}, \quad \begin{pmatrix}3, & 2, & 5\end{pmatrix}, \quad \begin{pmatrix}6, & -1, & 1\end{pmatrix}, \quad \begin{pmatrix}7, & 0, & 2\end{pmatrix}$

Solution.

1. The determinant
$$\begin{vmatrix}-3 & 0 & 4\\ 5 & -1 & 2\\ 1 & 1 & 3\end{vmatrix} = 39$$
and hence the vectors are linearly independent.
2. Four vectors in $\mathbb{R}^3$ are linearly dependent.

Question 21. In each part, determine whether the vectors are linearly independent or are linearly dependent in $\mathbb{R}^4$.

1. $\begin{pmatrix}3, & 8, & 7, & -3\end{pmatrix}, \begin{pmatrix}1, & 5, & 3, & -1\end{pmatrix}, \begin{pmatrix}2, & -1, & 2, & 6\end{pmatrix}, \begin{pmatrix}4, & 2, & 6, & 4\end{pmatrix}$
2. $\begin{pmatrix}3, & 0, & -3, & 6\end{pmatrix}, \begin{pmatrix}0, & 2, & 3, & 1\end{pmatrix}, \begin{pmatrix}0, & -2, & -2, & 0\end{pmatrix}, \begin{pmatrix}-2, & 1, & 2, & 1\end{pmatrix}$

Solution.

1. The determinant
$$\begin{vmatrix}3 & 8 & 7 & -3\\ 1 & 5 & 3 & -1\\ 2 & -1 & 2 & 6\\ 4 & 2 & 6 & 4\end{vmatrix} = 0$$
and hence the vectors are linearly dependent.
2. The determinant
$$\begin{vmatrix}3 & 0 & -3 & 6\\ 0 & 2 & 3 & 1\\ 0 & -2 & -2 & 0\\ -2 & 1 & 2 & 1\end{vmatrix} = 35 \neq 0$$
and hence the vectors are linearly independent.

Question 22. Prove the following theorem

Theorem 1 *$S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n\}$ spans $\mathbb{R}^n$ iff the determinant*

$$\begin{vmatrix}\boldsymbol{v}_1\\ \boldsymbol{v}_2\\ \vdots\\ \boldsymbol{v}_n\end{vmatrix} \neq 0.$$

Solution. S spans $\mathbb{R}^n$ iff the linear system

$$\alpha_1 \boldsymbol{v}_1 + \alpha_2 \boldsymbol{v}_2 + \cdots + \alpha_n \boldsymbol{v}_n = \boldsymbol{u}$$

is consistent for every $\boldsymbol{u} \in \mathbb{R}^n$, which is true iff the coefficient matrix

$$\begin{pmatrix} \boldsymbol{v}_1^\top & \boldsymbol{v}_2^\top & \cdots & \boldsymbol{v}_n^\top \end{pmatrix}$$

has nonzero determinant. Since the determinant of a square matrix is equal to the determinant of its transpose, it follows that S spans $\mathbb{R}^n$ iff

$$\begin{vmatrix} \boldsymbol{v}_1 \\ \boldsymbol{v}_2 \\ \vdots \\ \boldsymbol{v}_n \end{vmatrix} \neq 0.$$

Question 23. In each part, determine whether the matrices are linearly independent or dependent.

1. $\begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}$ in $\mathcal{M}_{2\times 2}$
2. $\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & -1 \\ -2 & -2 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ -2 & 2 \end{pmatrix}$ in $\mathcal{M}_{2\times 2}$
3. $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ in $\mathcal{M}_{2\times 3}$.

Solution.

1. Consider the equation

$$\alpha_1 \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} + \alpha_2 \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} + \alpha_3 \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

   Expanding both sides and equating the corresponding matrix entries, we obtain the system of linear equations

$$\begin{aligned} \alpha_1 + \alpha_2 &= 0 \\ 2\alpha_2 + \alpha_3 &= 0 \\ \alpha_1 + 2\alpha_2 + 2\alpha_3 &= 0 \\ 2\alpha_1 + \alpha_2 + \alpha_3 &= 0 \end{aligned}$$

   The augmented matrix of this linear system is

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 1 & 2 & 2 & 0 \\ 2 & 1 & 1 & 0 \end{array}\right)$$

Applying Gauss–Jordan elimination, we obtain the reduced row echelon form:

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

The system has only the trivial solution. It follows that the matrices are linearly independent in $\mathcal{M}_{2\times 2}$.

2. Consider the equation

$$\alpha_1 \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} 1 & -1 \\ -2 & -2 \end{pmatrix} + \alpha_3 \begin{pmatrix} 1 & 1 \\ -2 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Expanding both sides and equating the corresponding matrix entries, we obtain the system of linear equations

$$\begin{aligned} \alpha_1 + \alpha_2 + \alpha_3 &= 0 \\ -2\alpha_2 - 2\alpha_3 &= 0 \\ -\alpha_1 - \alpha_2 + \alpha_3 &= 0 \\ -2\alpha_2 + 2\alpha_3 &= 0 \end{aligned}$$

The augmented matrix of this linear system is

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -2 & -2 & 0 \\ -1 & -1 & 1 & 0 \\ 0 & -2 & 2 & 0 \end{array}\right)$$

Applying Gauss–Jordan elimination, we obtain the reduced row echelon form:

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

The system has only the trivial solution. It follows that the matrices are linearly independent in $\mathcal{M}_{2\times 2}$.

3. Consider the equation

$$\alpha_1 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} + \alpha_3 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Expanding both sides and equating the corresponding matrix entries, we obtain the system of linear equations

$$\begin{aligned} \alpha_1 &= 0 \\ \alpha_2 &= 0 \\ \alpha_3 &= 0 \end{aligned}$$

. Thus the system has only the trivial solution and it follows that the matrices are linearly independent in $\mathcal{M}_{2\times 3}$.

Question 24. Determine all values of a for which the following matrices are linearly independent in $\mathcal{M}_{2\times 2}$

$$\begin{pmatrix} 1 & 0 \\ 1 & a \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 \\ a & 1 \end{pmatrix}, \quad \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}$$

Solution. Consider the equation

$$\alpha_1 \begin{pmatrix} 1 & 0 \\ 1 & a \end{pmatrix} + \alpha_2 \begin{pmatrix} -1 & 0 \\ a & 1 \end{pmatrix} + \alpha_3 \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Expanding both sides and equating the corresponding matrix entries, we obtain the system of linear equations

$$\begin{aligned} \alpha_1 - \alpha_2 + 2\alpha_3 &= 0 \\ \alpha_1 + a\alpha_2 + \alpha_3 &= 0 \\ a\alpha_1 + \alpha_2 + 3\alpha_3 &= 0 \end{aligned}$$

The given matrices are linearly independent if and only if the system has only the trivial solution, which occurs only when the coefficient matrix has a nonzero determinant. Computing the determinant, we obtain

$$\begin{vmatrix} 1 & -1 & 2 \\ 1 & a & 1 \\ a & 1 & 3 \end{vmatrix} = \begin{vmatrix} 1 & -1 & 2 \\ 0 & a+1 & -1 \\ 0 & 1+a & 3-2a \end{vmatrix} = \begin{vmatrix} 1 & -1 & 2 \\ 0 & a+1 & -1 \\ 0 & 0 & 4-2a \end{vmatrix} = (a+1)(4-2a).$$

For the determinant to be nonzero, we require $a \neq -1, 2$. Thus the given matrices are linearly independent iff $a \neq -1, 2$.

Question 25. In each part, determine whether the three vectors lie in a plane in $\mathbb{R}^3$

1. $\boldsymbol{v}_1 = \begin{pmatrix} 2, & -2, & 0 \end{pmatrix}$, $\boldsymbol{v}_2 = \begin{pmatrix} 6, & 1, & 4 \end{pmatrix}$, $\boldsymbol{v}_3 = \begin{pmatrix} 2, & 0, & -4 \end{pmatrix}$
2. $\boldsymbol{v}_1 = \begin{pmatrix} -6, & 7, & 2 \end{pmatrix}$, $\boldsymbol{v}_2 = \begin{pmatrix} 3, & 2, & 4 \end{pmatrix}$, $\boldsymbol{v}_3 = \begin{pmatrix} 4, & -1, & 2 \end{pmatrix}$

Solution. To determine whether the three given vectors lie in a plane in $\mathbb{R}^3$, we check if they are linearly dependent. This is equivalent to computing the determinant of the matrix formed by taking the vectors as rows or columns. If the determinant is zero, the vectors are linearly dependent and lie in a plane; otherwise, they are linearly independent and span $\mathbb{R}^3$.

1. $$\begin{vmatrix} 2 & -2 & 0 \\ 6 & 1 & 4 \\ 2 & 0 & -4 \end{vmatrix} = \begin{vmatrix} 2 & -2 & 0 \\ 0 & 7 & 4 \\ 0 & 2 & -4 \end{vmatrix} = 2\begin{vmatrix} 7 & 4 \\ 2 & -4 \end{vmatrix} = 2 \times (-28 - 8) = -72 \neq 0$$

   Since the determinant is nonzero, the vectors are linearly independent and do not lie in a plane.

2. $$\begin{vmatrix} -6 & 7 & 2 \\ 3 & 2 & 4 \\ 4 & -1 & 2 \end{vmatrix} = (-6 \times 2 \times 2 + 7 \times 4 \times 4 - 6) - (16 + 24 + 42) = 0$$

   Since the determinant is zero, the vectors are linearly dependent and lie in a plane.

Question 26. In each part, determine whether the three vectors lie on the same line in $\mathbb{R}^3$

1. $\boldsymbol{v}_1 = \begin{pmatrix}-1, & 2, & 3\end{pmatrix}$, $\boldsymbol{v}_2 = \begin{pmatrix}-2, & -4, & -6\end{pmatrix}$, $\boldsymbol{v}_3 = \begin{pmatrix}-3, & 6, & 0\end{pmatrix}$
2. $\boldsymbol{v}_1 = \begin{pmatrix}2, & -1, & 4\end{pmatrix}$, $\boldsymbol{v}_2 = \begin{pmatrix}4, & 2, & 3\end{pmatrix}$, $\boldsymbol{v}_3 = \begin{pmatrix}2, & 7, & -6\end{pmatrix}$
3. $\boldsymbol{v}_1 = \begin{pmatrix}4, & 6, & 8\end{pmatrix}$, $\boldsymbol{v}_2 = \begin{pmatrix}2, & 3, & 4\end{pmatrix}$, $\boldsymbol{v}_3 = \begin{pmatrix}-2, & -3, & -4\end{pmatrix}$

Solution. Three vectors are collinear if and only if each vector is a scalar multiple of the other two. In other words, $\exists \alpha, \beta \in \mathbb{R}$ such that

$$\boldsymbol{v}_2 = \alpha \boldsymbol{v}_1, \quad \boldsymbol{v}_3 = \beta \boldsymbol{v}_1.$$

1. Suppose $\boldsymbol{v}_2 = \alpha \boldsymbol{v}_1$. Equating the corresponding components, we obtain

   $$-2 = -\alpha, \quad -4 = 2\alpha, \quad -6 = 3\alpha.$$

   Solving for α, we get

   $$\alpha = 2, \quad \alpha = -2, \quad \alpha = -2$$

   Since the values of α are inconsistent, no scalar α satisfies all three equations simultaneously. Hence, the vectors are not collinear.

2. Suppose $\boldsymbol{v}_2 = \alpha \boldsymbol{v}_1$. From the first two components, we obtain:

   $$4 = 2\alpha, \quad 2 = -\alpha.$$

   Solving for α, we get

   $$\alpha = 2, \quad \alpha = -2.$$

   The inconsistency between these values indicates that no single α satisfies both equations. Therefore, the vectors are not collinear.

3. Observing the relationships between the vectors, we find:

   $$\boldsymbol{v}_2 = 2\boldsymbol{v}_1, \quad \boldsymbol{v}_3 = -\frac{1}{2}\boldsymbol{v}_1.$$

   Since both $\boldsymbol{v}_2$ and $\boldsymbol{v}_3$ are scalar multiples of $\boldsymbol{v}_1$, the three vectors are collinear,

Question 27. For which values of λ do the following vectors form a linearly dependent set in $\mathbb{R}^3$?

$$\boldsymbol{v}_1 = \left(\lambda, \quad -\frac{1}{2}, \quad -\frac{1}{2}\right), \quad \boldsymbol{v}_2 = \left(-\frac{1}{2}, \quad \lambda, \quad -\frac{1}{2}\right), \quad \boldsymbol{v}_3 = \left(-\frac{1}{2}, \quad -\frac{1}{2}, \quad \lambda\right).$$

Solution. The three vectors are linearly dependent if and only if the following determinant is zero

$$\begin{vmatrix} \lambda & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \lambda & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \lambda \end{vmatrix}$$

Apply row operations, we get

$$\begin{vmatrix} \lambda & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \lambda & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \lambda \end{vmatrix} = \frac{1}{2^3}\begin{vmatrix} 2\lambda & -1 & -1 \\ -1 & 2\lambda & -1 \\ -1 & -1 & 2\lambda \end{vmatrix} = \frac{1}{2^3}\begin{vmatrix} 0 & -1+4\lambda^2 & -1-2\lambda \\ -1 & 2\lambda & -1 \\ 0 & -1-2\lambda & 2\lambda+1 \end{vmatrix}$$

$$= \frac{1}{2^3}\begin{vmatrix} 0 & 0 & -2-2\lambda+4\lambda^2 \\ -1 & 2\lambda & -1 \\ 0 & -1-2\lambda & 2\lambda+1 \end{vmatrix} = \frac{1}{2^3}\begin{vmatrix} -1 & 2\lambda & -1 \\ 0 & -1-2\lambda & 2\lambda+1 \\ 0 & 0 & -2-2\lambda+4\lambda^2 \end{vmatrix}$$

$$= \frac{1}{2^3}(2\lambda+1)(4\lambda^2-2\lambda-2)$$

The determinant is zero implies

$$(2\lambda+1)(4\lambda^2-2\lambda-2) = 0 \Longrightarrow \lambda = -\frac{1}{2}, 1$$

Question 28. For each part, first show that the vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ are linearly dependent in $\mathbb{R}^4$. Subsequently, demonstrate that each vector can be expressed as a linear combination of the remaining two.

1. $\boldsymbol{v}_1 = \begin{pmatrix}0, & 3, & 1, & -1\end{pmatrix}$, $\boldsymbol{v}_2 = \begin{pmatrix}6, & 0, & 5, & 1\end{pmatrix}$, $\boldsymbol{v}_3 = \begin{pmatrix}4, & -7, & 1, & 3\end{pmatrix}$
2. $\boldsymbol{v}_1 = \begin{pmatrix}1, & 2, & 3, & 4\end{pmatrix}$, $\boldsymbol{v}_2 = \begin{pmatrix}0, & 1, & 0, & -1\end{pmatrix}$, $\boldsymbol{v}_3 = \begin{pmatrix}1, & 3, & 3, & 3\end{pmatrix}$

Solution.

1. Consider the equation

$$\alpha_1\begin{pmatrix}0, & 3, & 1, & -1\end{pmatrix} + \alpha_2\begin{pmatrix}6, & 0, & 5, & 1\end{pmatrix} + \alpha_3\begin{pmatrix}4, & -7, & 1, & 3\end{pmatrix} = \begin{pmatrix}0, & 0, & 0, & 0\end{pmatrix}$$

Expanding both sides and equating the corresponding matrix entries, we obtain the system of linear equations

$$\begin{aligned} 6\alpha_2 + 4\alpha_3 &= 0 \\ 3\alpha_1 - 7\alpha_3 &= 0 \\ \alpha_1 + 5\alpha_2 + \alpha_3 &= 0 \\ -\alpha_1 + \alpha_2 + 3\alpha_3 &= 0 \end{aligned}$$

The augmented matrix of this linear system is

$$\left(\begin{array}{ccc|c} 0 & 6 & 4 & 0 \\ 3 & 0 & -7 & 0 \\ 1 & 5 & 1 & 0 \\ -1 & 1 & 3 & 0 \end{array}\right)$$

Applying Gauss–Jordan elimination, we obtain the reduced row echelon form:

$$\begin{pmatrix} 1 & 0 & -\frac{7}{3} & 0 \\ 0 & 1 & \frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Since the system has a free variable, the set $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ is linearly dependent. The solution set of the system is

$$\left\{ \begin{pmatrix} \frac{7t}{3}, & -\frac{2t}{3}, & t \end{pmatrix} \;\middle|\; t \in \mathbb{R} \right\}.$$

For example, setting $t = 3$ yields

$$7\boldsymbol{v}_1 - 2\boldsymbol{v}_2 + 3\boldsymbol{v}_3 = \boldsymbol{0},$$

confirming linear dependence. Furthermore, we can express each vector as a linear combination of the other two:

$$\boldsymbol{v}_1 = \frac{2}{7}\boldsymbol{v}_2 - \frac{3}{7}\boldsymbol{v}_3, \quad \boldsymbol{v}_2 = \frac{7}{2}\boldsymbol{v}_1 + \frac{3}{2}\boldsymbol{v}_3, \quad \boldsymbol{v}_3 = -\frac{7}{3}\boldsymbol{v}_1 + \frac{2}{3}\boldsymbol{v}_2.$$

2. Consider the equation

$$\alpha_1 \begin{pmatrix} 1, & 2, & 3, & 4 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0, & 1, & 0, & -1 \end{pmatrix} + \alpha_3 \begin{pmatrix} 1, & 3, & 3, & 3 \end{pmatrix} = \begin{pmatrix} 0, & 0, & 0, & 0 \end{pmatrix}$$

Expanding both sides and equating the corresponding matrix entries, we obtain the system of linear equations

$$\begin{aligned} \alpha_1 + \alpha_3 &= 0 \\ 2\alpha_1 + \alpha_2 + 3\alpha_3 &= 0 \\ 3\alpha_1 + 3\alpha_3 &= 0 \\ 4\alpha_1 - \alpha_2 + 3\alpha_3 &= 0 \end{aligned}$$

The augmented matrix of this linear system is

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 2 & 1 & 3 & 0 \\ 3 & 0 & 3 & 0 \\ 4 & -1 & 3 & 0 \end{array}\right)$$

Applying Gauss–Jordan elimination, we obtain the reduced row echelon form:

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Since the system has a free variable, the set $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ is linearly dependent. The solution set of the system is

$$\left\{ \begin{pmatrix} -t, & -t, & t \end{pmatrix} \;\middle|\; t \in \mathbb{R} \right\}.$$

For example, setting $t = 1$ yields

$$-\boldsymbol{v}_1 - \boldsymbol{v}_2 + \boldsymbol{v}_3 = \boldsymbol{0},$$

confirming linear dependence. Furthermore, we can express each vector as a linear combination of the other two:

$$\boldsymbol{v}_1 = -\boldsymbol{v}_2 + \boldsymbol{v}_3, \quad \boldsymbol{v}_2 = -\boldsymbol{v}_1 + \boldsymbol{v}_3, \quad \boldsymbol{v}_3 = \boldsymbol{v}_1 + \boldsymbol{v}_2.$$