# Tutorial 4

# Determinants

**Question 1.** Compute the determinants of the following matrices using the standard determinant formula for $2 \times 2$ matrices and Sarrus' rule for $3 \times 3$ matrices.

1. $\begin{pmatrix} 2 & 5 \\ 3 & 1 \end{pmatrix}$
2. $\begin{pmatrix} 5 & -3 \\ 2 & 0 \end{pmatrix}$
3. $\begin{pmatrix} 6 & -12 \\ -4 & 8 \end{pmatrix}$
4. $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$
5. $\begin{pmatrix} 2 & 0 & 5 \\ -4 & 1 & 7 \\ 0 & 3 & -3 \end{pmatrix}$
6. $\begin{pmatrix} 3 & -2 & 4 \\ 5 & 1 & -2 \\ -1 & 3 & 6 \end{pmatrix}$
7. $\begin{pmatrix} 5 & 0 & 0 \\ 3 & -2 & 0 \\ -1 & 8 & 4 \end{pmatrix}$
8. $\begin{pmatrix} -6 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{pmatrix}$
9. $\begin{pmatrix} 3 & 1 & -2 \\ -1 & 4 & 5 \\ 3 & 1 & -2 \end{pmatrix}$
10. $\begin{pmatrix} \alpha-3 & 5 \\ -3 & \alpha-2 \end{pmatrix}$
11. $\begin{pmatrix} -1 & 1 & 2 \\ 3 & 0 & -5 \\ 1 & 7 & 2 \end{pmatrix}$
12. $\begin{pmatrix} -2 & 7 & 6 \\ 5 & 1 & -2 \\ 3 & 8 & 4 \end{pmatrix}$
13. $\begin{pmatrix} 3 & 0 & 0 \\ 2 & -1 & 5 \\ 1 & 9 & -4 \end{pmatrix}$
14. $\begin{pmatrix} -2 & 1 & 4 \\ 3 & 5 & -7 \\ 1 & 6 & 2 \end{pmatrix}$
15. $\begin{pmatrix} \beta & -4 & 3 \\ 2 & 1 & \beta^2 \\ 4 & \beta-1 & 2 \end{pmatrix}$

*Solution.*

1. $\begin{vmatrix} 2 & 5 \\ 3 & 1 \end{vmatrix} = 2 \times 1 - 3 \times 5 = -13$
2. $6$
3. $0$
4. $\cos^2\theta + \sin^2\theta = 1$
5. $2 \times 1 \times (-3) + 0 + 5 \times (-4) \times 3 - 0 - 2 \times 3 \times 7 - 0 = -6 - 60 - 42 = -108$
6. $156$
7. $-40$
8. $156$
9. $0$
10. $\alpha^2 - 5\alpha + 21$
11. $-4$
12. $0$
13. $-123$
14. $-65$
15. $-\beta^4 + \beta^3 - 16\beta^2 + 8\beta - 2$

**Question 2.** Find all the minors and cofactors of the matrix $A$.

1. $A = \begin{pmatrix} 1 & -2 & 3 \\ 6 & 7 & -1 \\ -3 & 1 & 4 \end{pmatrix}$, 2. $A = \begin{pmatrix} 1 & 1 & 2 \\ 3 & 3 & 6 \\ 0 & 1 & 4 \end{pmatrix}$

*Solution.*

1. Minors

$$M_{11} = \begin{vmatrix} 7 & -1 \\ 1 & 4 \end{vmatrix} = 29, \quad M_{12} = 21, \quad M_{13} = 27, \quad M_{21} = -11, \quad M_{22} = 13,$$

$$M_{23} = -5, \quad M_{31} = -19, \quad M_{32} = 19, \quad M_{33} = 19.$$

Cofactors:

$$C_{11} = (-1)^{1+1} \times 29 = 29, \quad C_{12} = -21, \quad C_{13} = 27, \quad C_{21} = 11, \quad C_{22} = 13, \quad C_{23} = 5,$$

$$C_{31} = -19, \quad C_{32} = 19, \quad C_{33} = 19$$

**Question 3.** Let

$$A = \begin{pmatrix} 2 & 3 & -1 & 1 \\ -3 & 2 & 0 & 3 \\ 3 & -2 & 1 & 0 \\ 3 & -1 & 1 & 4 \end{pmatrix}$$

Find

1. Minor $M_{32}$ and cofactor $C_{32}$
2. Minor $M_{41}$ and cofactor $C_{41}$
3. Minor $M_{44}$ and cofactor $C_{44}$
4. Minor $M_{24}$ and cofactor $C_{24}$

*Solution.*

1. $$M_{32} = \begin{vmatrix} 2 & -1 & 1 \\ -3 & 0 & 3 \\ 3 & 1 & 4 \end{vmatrix} = 3 \times \begin{vmatrix} -1 & 1 \\ 1 & 4 \end{vmatrix} - 3 \times \begin{vmatrix} 2 & -1 \\ 3 & 1 \end{vmatrix} = 3 \times (-5) - 3 \times 5 = -30, \quad C_{32} = (-1)^{3+2} M_{32} = 30$$
2. $M_{41} = -1$, $C_{41} = 1$
3. $M_{44} = 13$, $C_{44} = 13$
4. $M_{24} = -5$, $C_{24} = -5$

**Question 4.** For each of the given matrices $A, B, C$, find the specified minors and cofactors.

$$A = \begin{pmatrix} 2 & 4 & 3 \\ 3 & -1 & 6 \\ 5 & -2 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 2 & -3 & 1 \\ 1 & 4 & 2 & -1 \\ 3 & -2 & 4 & 0 \\ 4 & -1 & 1 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} -3 & 3 & 0 & 5 \\ 2 & 1 & -1 & 4 \\ 6 & -3 & 4 & 0 \\ -1 & 5 & 1 & -2 \end{pmatrix}$$

1. Minors $M_{12}, M_{21}, M_{23}, M_{31}$ and cofactors $C_{12}, C_{22}, C_{32}, C_{21}$ for matrix $A$.
2. Minors $M_{12}, M_{42}, M_{24}, M_{31}, M_{34}$ and cofactors $C_{12}, C_{24}, C_{31}, C_{42}, C_{33}$ for matrix $B$.
3. Minors $M_{11}, M_{41}, M_{23}, M_{32}, M_{44}$ and cofactors $C_{12}, C_{24}, C_{31}, C_{42}, C_{33}$ for matrix $C$.

*Solution.*

1.
$$M_{12} = -18, \quad M_{21} = 22, \quad M_{23} = -24, \quad M_{31} = 27,$$
$$C_{12} = 18, \quad C_{22} = -7, \quad C_{32} = -3, \quad C_{21} = 27$$

2.
$$M_{12} = 13, \quad M_{42} = 7, \quad M_{24} = 11, \quad M_{31} = 5, \quad M_{34} = 65$$
$$C_{12} = -13 \quad , C_{24} = 11, \quad C_{31} = 5, \quad C_{42} = 7, \quad C_{33} = -25$$

3.
$$M_{11} = -94, \quad M_{41} = -43, \quad M_{23} = 153, \quad M_{32} = 11, \quad M_{44} = -45$$
$$C_{12} = 74, \quad C_{24} = 39, \quad C_{31} = 24, \quad C_{42} = 118, \quad C_{33} = 121$$

**Question 5.** Find the minors $M_{11}, M_{31}, M_{23}$ and cofactors $C_{12}, C_{32}, C_{22}$ for the following matrix

$$A = \begin{pmatrix} \alpha+1 & \alpha & \alpha-7 \\ \alpha-4 & \alpha+5 & \alpha-3 \\ \alpha-1 & \alpha & \alpha+2 \end{pmatrix}$$

*Solution.*

$$M_{11} = \begin{vmatrix} \alpha+5 & \alpha-3 \\ \alpha & \alpha+2 \end{vmatrix} = (\alpha+2)(\alpha+5) - \alpha(\alpha-3) = 7\alpha+10+3\alpha = 10\alpha+10, \quad M_{31} = 35-\alpha, \quad M_{23} = 2\alpha.$$

$$C_{12} = 11 - 2\alpha, \quad C_{32} = 31 - 9\alpha, \quad C_{22} = 11\alpha - 5.$$

**Question 6.** Compute the determinant of the given matrix. If the matrix is invertible, find its inverse using the formula for $2 \times 2$ matrices.

1. $\begin{pmatrix} 3 & 5 \\ -2 & 4 \end{pmatrix}$
2. $\begin{pmatrix} 4 & 1 \\ 8 & 2 \end{pmatrix}$
3. $\begin{pmatrix} -5 & 7 \\ -7 & -2 \end{pmatrix}$
4. $\begin{pmatrix} \sqrt{2} & \sqrt{6} \\ 4 & \sqrt{3} \end{pmatrix}$

*Solution.*

1. The determinant of the matrix is $3 \times 4 - 5 \times (-2) - 12 + 10 = 22$. The inverse of the matrix is
$$\frac{1}{22}\begin{pmatrix} 4 & -5 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} \frac{2}{11} & -\frac{5}{22} \\ \frac{1}{11} & \frac{3}{22} \end{pmatrix}.$$

2. The matrix is not invertible
$$\begin{vmatrix} 4 & 1 \\ 8 & 2 \end{vmatrix} = 2 \times 4 - 8 \times 1 = 0$$

3.

$$\begin{vmatrix} -5 & 7 \\ -7 & -2 \end{vmatrix} = 59, \quad \begin{pmatrix} -5 & 7 \\ -7 & -2 \end{pmatrix}^{-1} = \begin{pmatrix} -\frac{2}{59} & -\frac{7}{59} \\ \frac{7}{59} & -\frac{5}{59} \end{pmatrix}$$

4.

$$\begin{vmatrix} \sqrt{2} & \sqrt{6} \\ 4 & \sqrt{3} \end{vmatrix} = -3\sqrt{6}, \quad \begin{pmatrix} \sqrt{2} & \sqrt{6} \\ 4 & \sqrt{3} \end{pmatrix}^{-1} = \begin{pmatrix} -\frac{\sqrt{2}}{6} & \frac{1}{3} \\ \frac{2\sqrt{6}}{9} & -\frac{\sqrt{3}}{9} \end{pmatrix}$$

**Question 7.** Find all values of $\lambda$ for which $\det(A) = 0$.

1. $A = \begin{pmatrix} \lambda - 2 & 1 \\ -5 & \lambda + 4 \end{pmatrix}$
2. $A = \begin{pmatrix} \lambda - 1 & 0 \\ 2 & \lambda + 1 \end{pmatrix}$
3. $A = \begin{pmatrix} \lambda - 4 & 0 & 0 \\ 0 & \lambda & 2 \\ 0 & 3 & \lambda - 1 \end{pmatrix}$
4. $A = \begin{pmatrix} \lambda - 4 & 4 & 0 \\ -1 & \lambda & 0 \\ 0 & 0 & \lambda - 5 \end{pmatrix}$
5. $A = \begin{pmatrix} \lambda & 2 \\ 5 & \lambda + 3 \end{pmatrix}$
6. $A = \begin{pmatrix} 15 & \lambda - 4 \\ \lambda + 7 & -2 \end{pmatrix}$
7. $A = \begin{pmatrix} \lambda - 3 & 5 & -19 \\ 0 & \lambda - 1 & 6 \\ 0 & 0 & \lambda - 2 \end{pmatrix}$

*Solution.*

1.

$$\det(A) = (\lambda - 2)(\lambda + 4) + 5 = \lambda^2 + 2\lambda - 3, \quad \det(A) = 0 \iff \lambda = 1, 3$$

2.

$$\det(A) = (\lambda + 1)(\lambda - 1), \quad \det(A) = 0 \iff \lambda = 1, -1$$

3.

$$\det(A) = (\lambda - 4) \begin{vmatrix} \lambda & 2 \\ 3 & \lambda - 1 \end{vmatrix} = (\lambda - 4)(\lambda^2 - \lambda - 6), \quad \det(A) = 0 \iff \lambda = -2,\ 3,\ 4$$

4.

$$\det(A) = (\lambda - 5) \begin{vmatrix} \lambda - 4 & 4 \\ -1 & \lambda \end{vmatrix} = (\lambda - 5)(\lambda^2 - 4\lambda + 4), \quad \det(A) = 0 \iff \lambda = -2,\ 5$$

5.

$$\det(A) = \lambda(\lambda + 3) - 10 = \lambda^2 + 3\lambda - 10, \quad \det(A) = 0 \iff \lambda = -5, 2$$

6.

$$\det(A) = -\lambda^2 - 3\lambda - 2, \quad \det(A) = 0 \iff \lambda = -2, -1$$

7.
$$\det(A) = (\lambda-3)(\lambda-1)(\lambda-2), \quad \det(A)=0 \Longleftrightarrow \lambda = 1,2,3$$

**Question 8.** Let
$$A = \begin{pmatrix} 3 & -2 & 4 \\ 5 & 1 & -2 \\ -1 & 3 & 6 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 1 & 2 \\ 3 & 0 & -5 \\ 1 & 7 & 2 \end{pmatrix}.$$
Compute the determinant of matrices $A, B$ by a cofactor expansion along

1. the first row
2. the second row
3. the third row
4. the first column
5. the second column
6. the third column

*Solution.*

1. Along the first row
$$\begin{aligned}\det(A) &= 3\begin{vmatrix}1 & -2\\ 3 & 6\end{vmatrix} + 2\begin{vmatrix}5 & -2\\ -1 & 6\end{vmatrix} + 4\begin{vmatrix}5 & 1\\ -1 & 3\end{vmatrix} = 3\times 12 + 2\times 28 + 4\times 16 = 156.\\ \det(B) &= -\begin{vmatrix}0 & -5\\ 7 & 2\end{vmatrix} - \begin{vmatrix}3 & -5\\ 1 & 2\end{vmatrix} + 2\begin{vmatrix}3 & 0\\ 1 & 7\end{vmatrix} = -35 - 11 + 42 = -4.\end{aligned}$$

2. Along the second row
$$\begin{aligned}\det(A) &= 5\times 24 + 1\times 22 + (-2)\times(-7) = 156.\\ \det(B) &= 3\times 12 + 0 + (-5)\times 8 = -4.\end{aligned}$$

3. Along the third row
$$\begin{aligned}\det(A) &= (-1)\times 0 + 3\times 26 + 6\times 13 = 156.\\ \det(B) &= 1\times(-5) + 7\times 1 + 2\times(-3) = -4.\end{aligned}$$

4. Along the first column
$$\begin{aligned}\det(A) &= 3\times 12 + 5\times 24 + (-1)\times 0 = 156.\\ \det(B) &= (-1)\times 35 + 3\times 12 + 1\times(-5) = -4.\end{aligned}$$

5. Along the second column
$$\begin{aligned}\det(A) &= (-2)\times(-28) + 1\times 22 + 3\times 26 = 156.\\ \det(B) &= 1\times(-11) + 0 + 7\times 1 = -4.\end{aligned}$$

6. Along the third column
$$\begin{aligned}\det(A) &= 4\times 16 + (-2)\times(-7) + 6\times 13 = 156.\\ \det(B) &= 2\times 21 + (-5)\times 8 + 2\times(-3) = -4.\end{aligned}$$

**Question 9.** Compute the determinant of the following matrices by a cofactor expansion along a suitable row or column

1. $\begin{pmatrix} -3 & 0 & 7 \\ 2 & 5 & 1 \\ -1 & 0 & 5 \end{pmatrix}$

2. $\begin{pmatrix} 3 & 3 & 1 \\ 1 & 0 & -4 \\ 1 & -3 & 5 \end{pmatrix}$

3. $\begin{pmatrix} k+1 & k-1 & 7 \\ 2 & k-3 & 4 \\ 5 & k+1 & k \end{pmatrix}$

4. $\begin{pmatrix} 5 & 2 & 1 & 0 \\ -1 & 3 & 5 & 2 \\ 4 & 1 & 0 & 2 \\ 0 & 2 & 3 & 0 \end{pmatrix}$

5. $\begin{pmatrix} 0 & 5 & 4 & 0 \\ 4 & 1 & -2 & 7 \\ -1 & 0 & 3 & 0 \\ 0 & 2 & 1 & 5 \end{pmatrix}$

6. $\begin{pmatrix} 2 & 1 & 9 & 7 \\ 0 & -1 & 3 & 8 \\ 0 & 0 & 5 & 2 \\ 0 & 0 & 0 & 6 \end{pmatrix}$

7. $\begin{pmatrix} 3 & 3 & 0 & 5 \\ 2 & -2 & 0 & -2 \\ 4 & 1 & -3 & 0 \\ 2 & 10 & 3 & 2 \end{pmatrix}$

8. $\begin{pmatrix} 0 & 4 & 1 & 3 & -2 \\ 2 & 2 & 3 & -1 & 0 \\ 3 & 1 & 2 & -5 & 1 \\ 1 & 0 & -4 & 0 & 0 \\ 0 & 3 & 0 & 0 & 2 \end{pmatrix}$

9. $\begin{pmatrix} 0 & 2 & 3 & 4 & -1 \\ 0 & 1 & 0 & 0 & -3 \\ 1 & 4 & 2 & 0 & -3 \\ 1 & 0 & 2 & 1 & 2 \\ 0 & 0 & 1 & -2 & 0 \end{pmatrix}$

*Solution.*

1. Along the second column

$$\begin{vmatrix} -3 & 0 & 7 \\ 2 & 5 & 1 \\ -1 & 0 & 5 \end{vmatrix} = 5 \begin{vmatrix} -3 & 7 \\ -1 & 5 \end{vmatrix} = 5 \times (-8) = -40$$

2. Along the second row

$$1 \times (-18) + 0 + (-4) \times 12 = -66$$

3. Along the third column

$$7(17 - 3k) + 4(-k^2 + 3k - 6) + k(k^2 - 4k - 1) = k^3 - 8k^2 - 10k + 95$$

4. Along the fourth column

$$0 + 2 \times (-1) + 2 \times 1 + 0 = 0$$

5. Along the fourth column

$$0 + 7 \times (-3) + 0 + 5 \times (-46) = -251$$

6. Along the fourth row

$$6 \times (-10) = -60$$

7. Along the third column

$$0 + 0 + (-3) \times 144 + 3 \times (-32) = -528$$

8. Along the fourth row

$$1 \times 128 + 0 + (-4) \times (-56) + 0 = 352$$

9. Along the second row

$$1 \times (-51) + (-3) \times (-42) = 75$$

**Question 10.** Evaluate the determinant of the given matrix by inspection.

1. $\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

2. $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

3. $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 4 & 3 & 0 \\ 1 & 2 & 3 & 8 \end{pmatrix}$

4. $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 4 \end{pmatrix}$

5. $\begin{pmatrix} 1 & 2 & 7 & -3 \\ 0 & 1 & -4 & 1 \\ 0 & 0 & 2 & 7 \\ 0 & 0 & 0 & 3 \end{pmatrix}$

6. $\begin{pmatrix} -3 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 40 & 10 & -1 & 0 \\ 100 & 200 & -23 & 3 \end{pmatrix}$

*Solution.*

1. $-1$, 2. 8, 3. 0, 4. 24, 5. 6, 6. 18

**Question 11.** For each matrix, show that the value of the determinant is independent of $\theta$.

1. $\begin{pmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{pmatrix}$

2. $\begin{pmatrix} \sin\theta & \cos\theta & 0 \\ -\cos\theta & \sin\theta & 0 \\ \sin\theta - \cos\theta & \sin\theta + \cos\theta & 1 \end{pmatrix}$

*Solution.*

1.

$$\begin{vmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{vmatrix} = \sin^2\theta + \cos^2\theta = 1.$$

2.

$$\begin{vmatrix} \sin\theta & \cos\theta & 0 \\ -\cos\theta & \sin\theta & 0 \\ \sin\theta - \cos\theta & \sin\theta + \cos\theta & 1 \end{vmatrix} = \begin{vmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{vmatrix} = 1$$

**Question 12.** By inspection, what is the relationship between the following determinants?

$$d_1 = \begin{vmatrix} a & b & c \\ d & 1 & f \\ g & 0 & 1 \end{vmatrix}, \quad d_2 = \begin{vmatrix} a+\lambda & b & c \\ d & 1 & f \\ g & 0 & 1 \end{vmatrix}$$

*Solution.* Let $C_{12}, C_{13}$ denote the cofactors for the first matrix corresponding to entry $(1,2)$ and $1,3)$, i.e. $b$ and $c$. Then

$$d_1 = a + bC_{12} + cC_{13},\ \ d_2 = (a+\lambda) + bC_{12} + cC_{13}.$$

Thus

$$d_2 = d_1 + \lambda.$$

**Question 13.** Show that

$$\det(A) = \frac{1}{2}\begin{vmatrix} \mathrm{tr}(A) & 1 \\ \mathrm{tr}(A^2) & \mathrm{tr}(A) \end{vmatrix}$$

for every matrix $A \in \mathcal{M}_{2\times 2}$.

*Solution.* Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Then $\det(A) = ad - bc$, and

$$A^2 = \begin{pmatrix} a^2+bc & ab+bd \\ ac+dc & cd+d^2 \end{pmatrix} \Longrightarrow \mathrm{tr}\left(A^2\right) = a^2+bc+cd+d^2 = a^2+d^2+2bc$$

We have

$$\begin{aligned} \frac{1}{2}\begin{vmatrix} \mathrm{tr}(A) & 1 \\ \mathrm{tr}(A^2) & \mathrm{tr}(A) \end{vmatrix} &= \frac{1}{2} = \frac{1}{2}\left(\mathrm{tr}\,(A)^2 - \mathrm{tr}\left(A^2\right)\right) = \frac{1}{2}\left((a+d)^2 - (a^2+d^2+2bc)\right) \\ &= \frac{1}{2}(2ad - 2bc) = ad - bc = \det(A). \end{aligned}$$

**Question 14.** What can you say about $\det(A)$, where $A \in \mathcal{M}_{n\times n}$ has entries all equal to 1?

*Solution.*
For $n = 1$, $\det(A) = 1$.
For $n \geq 2$, the matrix $A$ has all rows identical, hence $\det(A) = 0$.

**Question 15.** What is the maximum number of zeros that a $3 \times 3$ matrix can have without having a zero determinant? Why?

*Solution.* We note that $I_3$ has 6 zero entries and nonzero determinant.

If a $3\times 3$ matrix $A$ has more than 6 zero entries, it must have one row or one column entirely zero, making its determinant 0.

Hence the answer is 6.

**Question 16.** Explain why the determinant of a square matrix with integer entries must be an integer.

*Solution.* If we consider the cofactor expansion for computing the determinant, the determinant is always a sum of products of matrix entries and their minors (which are themselves determinants of smaller matrices).

The determinant calculation involves addition, subtraction, and multiplication, all of which preserve integer values when applied to integers.

**Question 17.** Prove that the three points $(x_1, y_1)$, $(x_2, y_2)$, and $(x_3, y_3)$ are collinear if and only if

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

*Solution.* Three points $(x_1, y_1)$, $(x_2, y_2)$, and $(x_3, y_3)$ are collinear if they lie on the same straight line. This is equivalent to

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{y_3 - y_1}{x_3 - x_1} \iff (x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1) = 0,$$

which is equivalent to

$$x_2y_3 - x_3y_2 - x_1y_3 + x_3y_1 + x_1y_2 - x_2y_1 = 0.$$

Compute the given determinant by cofactor expansion along the third column gives

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix} - \begin{vmatrix} x_1 & y_1 \\ x_3 & y_3 \end{vmatrix} + \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} = x_2y_3 - x_3y_2 - x_1y_3 + x_3y_1 + x_1y_2 - x_2y_1.$$

**Question 18.** Prove that the equation of the line through the distinct points $(a_1, b_1)$ and $(a_2, b_2)$ can be written as

$$\begin{vmatrix} x & y & 1 \\ a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \end{vmatrix} = 0$$

*Solution.* The line through $(a_1, b_1)$ and $(a_2, b_2)$ is

$$(a_1 - a_2)y + (b_2 - b_1)x = a_1b_2 - a_2b_1.$$

Cofactor expansion along the first row for computing the determinant gives

$$\begin{vmatrix} x & y & 1 \\ a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \end{vmatrix} = x\begin{vmatrix} b_1 & 1 \\ b_2 & 1 \end{vmatrix} - y\begin{vmatrix} a_1 & 1 \\ a_2 & 1 \end{vmatrix} + \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = x(b_1 - b_2) - y(a_1 - a_2) + (a_1b_2 - a_2b_1) = 0$$

i.e.

$$(a_1 - a_2)y + (b_2 - b_1)x = a_1b_2 - a_2b_1.$$

**Question 19.** Prove

**Theorem 1** *A triangular matrix is invertible if and only if its diagonal entries are all nonzero.*

*Solution.* Let $A = (a_{ij} \in \mathcal{M}_{n\times n}$ be a triangular matrix, then its determinant is given by

$$\det(A) = a_{11}a_{22}\cdots a_{nn}$$

We have proved during the lecture that a matrix is invertible if and only if its determinant is nonzero. Thus $A$ is invertible iff the diagonal entries are all nonzero.

**Question 20.** Prove Theorem 3 using Theorems 1 and 2.

**Theorem 2** *The inverse of an invertible lower triangular matrix is lower triangular, and the inverse of an invertible upper triangular matrix is upper triangular.*

**Theorem 3** *If $A$ is an invertible matrix, then*

$$A^{-1} = \frac{1}{\det(A)}\operatorname{adj}(A).$$

*Solution.* Assume $A \in \mathcal{M}_{n\times n}$ is upper triangular and invertible. Since

$$A^{-1} = \frac{1}{\det(A)}\operatorname{adj}(A)$$

we can prove that $A^{-1}$ is upper triangular by showing that $\operatorname{adj}(A)$ is upper triangular or, equivalently, that the matrix of cofactors is lower triangular.

To show that the matrix of cofactors of $A$ is lower triangular, we show that every cofactor $C_{ij} = 0$ for $i < j$. Since

$$C_{ij} = (-1)^{i+j}M_{ij},$$

it suffices to show $M_{ij} = 0$ for $i < j$.

Let $B_{ij}$ be the matrix that results when the $i$th row and $j$th column of $A$ are deleted, where $i < j$, then

$$M_{ij} = \det(B_{ij}).$$

Since $A \in \mathcal{M}_{n\times n}$ is upper triangular and $i < j$, $B_{ij}$ is upper triangular. Furthermore, the $i$th row of $B_{ij}$ corresponds to the $(i+1)$th row of $A$ which starts with at least $i$ zeros, none of those zeros were deleted because $j > i$, making the $(i, i)$-entry of $B_{ij}$ to be zero. And by Theorem 1, $M_{ij} = \det(B_{ij}) = 0$.

The proof for lower triangular matrix is similar.

**Question 21.** With the following examples, verify that $\det(A) = \det(A^\top)$

1. $A = \begin{pmatrix} -2 & 3 \\ 1 & 4 \end{pmatrix}$
2. $A = \begin{pmatrix} -6 & 1 \\ 2 & -2 \end{pmatrix}$
3. $A = \begin{pmatrix} 2 & -1 & 3 \\ 1 & 2 & 4 \\ 5 & -3 & 6 \end{pmatrix}$
4. $A = \begin{pmatrix} 4 & 2 & -1 \\ 0 & 2 & -3 \\ -1 & 1 & 5 \end{pmatrix}$

*Solution.*

1. $\det(A) = \det(A^\top) = -11$
2. $\det(A) = \det(A^\top) = 10$
3. $\det(A) = \det(A^\top) = -5$
4. $\det(A) = \det(A^\top) = 56$

**Question 22.** Each of the following matrices is obtained from the identity matrix by performing a single elementary row operation. Identify the specific operation performed and find the determinant of each matrix using the properties of determinants in relation to row operations.

1. $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$
2. $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$
3. $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -5 & 0 & 1 \end{pmatrix}$
4. $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{1}{3} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$
5. $\begin{pmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
6. $\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
7. $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
8. $\begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
9. $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -4 \end{pmatrix}$
10. $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

*Solution.*

1. This elementary matrix is obtained by multiplying the third row of $I_4$ by $-5$, which, according to determinant properties, results in a determinant of $-5$
2. The elementary matrix is obtained by interchanging the second and third rows of $I_4$, which results in a determinant of $-1$.
3. The elementary matrix is obtained by adding $-5$ times the first row to the third row of $I_3$, yielding a determinant of 1.
4. The elementary matrix is obtained by multiplying the second row of $I_4$ by $-\frac{1}{3}$, which gives a determinant of $-\frac{1}{3}$.
5. $R_1 \to R_1 - 3R_2$, determinant $= 1$.
6. $R_2 \to R_2 + 2R_1$, determinant $= 1$.
7. $R_1 \to 0R_1$, determinant $= 0$.
8. $R_1 \to \frac{1}{2}R_1$, determinant $= \frac{1}{2}$.

9. $R_3 \to -4R_3$, determinant $= -4$.
10. $R_1 \leftrightarrow R_2$, determinant $= -1$.

**Question 23.** Find the determinant of the following matrices using some combination of row operations, column operations, and cofactor expansion.

1. $\begin{pmatrix} 10 & 4 & 21 \\ 0 & -4 & 3 \\ -5 & -1 & -12 \end{pmatrix}$
2. $\begin{pmatrix} 18 & -9 & -14 \\ 6 & -3 & -5 \\ -3 & 1 & 2 \end{pmatrix}$
3. $\begin{pmatrix} 3 & -6 & 9 \\ -2 & 7 & -2 \\ 0 & 1 & 5 \end{pmatrix}$
4. $\begin{pmatrix} 3 & 6 & -9 \\ 0 & 0 & -2 \\ -2 & 1 & 5 \end{pmatrix}$
5. $\begin{pmatrix} 1 & -3 & 0 \\ -2 & 4 & 1 \\ 5 & -2 & 2 \end{pmatrix}$
6. $\begin{pmatrix} 1 & 3 & 1 & 5 & 3 \\ -2 & -7 & 0 & -4 & 2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$
7. $\begin{pmatrix} 1 & -2 & 3 & 1 \\ 5 & -9 & 6 & 3 \\ -1 & 2 & -6 & -2 \\ 2 & 8 & 6 & 1 \end{pmatrix}$
8. $\begin{pmatrix} -8 & 4 & -3 & 2 \\ 2 & 1 & -1 & -1 \\ -3 & -5 & 4 & 0 \\ 2 & -4 & 3 & -1 \end{pmatrix}$
9. $\begin{pmatrix} 1 & -1 & 5 & 1 \\ -2 & 1 & -7 & 1 \\ -3 & 2 & -12 & -2 \\ 2 & -1 & 9 & 1 \end{pmatrix}$
10. $\begin{pmatrix} 5 & 3 & -8 & 4 \\ \frac{15}{2} & \frac{1}{2} & -1 & -7 \\ -\frac{5}{2} & \frac{3}{2} & -4 & 1 \\ 10 & -3 & 8 & -8 \end{pmatrix}$
11. $\begin{pmatrix} 1 & 2 & -1 & 3 & 0 \\ 2 & 4 & -3 & 1 & -4 \\ 2 & 6 & 4 & 8 & -4 \\ -3 & -8 & -1 & 1 & 0 \\ 1 & 3 & 3 & 10 & 1 \end{pmatrix}$
12. $\begin{pmatrix} -4 & 1 & 1 & 1 & 1 \\ 1 & -4 & 1 & 1 & 1 \\ 1 & 1 & -4 & 1 & 1 \\ 1 & 1 & 1 & -4 & 1 \\ 1 & 1 & 1 & 1 & -4 \end{pmatrix}$

*Solution.*

1.
$$\begin{vmatrix} 10 & 4 & 21 \\ 0 & -4 & 3 \\ -5 & -1 & -12 \end{vmatrix} \overset{R_3\to R_3+\frac{1}{2}R_1}{=} \begin{vmatrix} 10 & 4 & 21 \\ 0 & -4 & 3 \\ 0 & 1 & -\frac{3}{2} \end{vmatrix} \overset{R_2\to R_2+4R_3}{=} \begin{vmatrix} 10 & 4 & 21 \\ 0 & 0 & -3 \\ 0 & 1 & -\frac{3}{2} \end{vmatrix} \overset{R_2\leftrightarrow R_3}{=} -\begin{vmatrix} 10 & 4 & 21 \\ 0 & 1 & -\frac{3}{2} \\ 0 & 0 & -3 \end{vmatrix} = 30$$

2.
$$\begin{vmatrix} 18 & -9 & -14 \\ 6 & -3 & -5 \\ -3 & 1 & 2 \end{vmatrix} \overset{R_1\to R_1-3R_2}{=} \begin{vmatrix} 0 & 0 & 1 \\ 6 & -3 & -5 \\ -3 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 6 & -3 \\ -3 & 1 \end{vmatrix} = 6-9=-3$$

3.
$$\begin{vmatrix} 3 & -6 & 9 \\ -2 & 7 & -2 \\ 0 & 1 & 5 \end{vmatrix} \overset{R_1\to R_1+6R_3,\ R_2\to R_2-7R_3}{=} \begin{vmatrix} 3 & 0 & 39 \\ -2 & 0 & -37 \\ 0 & 1 & 5 \end{vmatrix} = -\begin{vmatrix} 3 & 39 \\ -2 & -37 \end{vmatrix} = 33$$

4.
$$\begin{vmatrix} 3 & 6 & -9 \\ 0 & 0 & -2 \\ -2 & 1 & 5 \end{vmatrix} = 2\begin{vmatrix} 3 & 6 \\ -2 & 1 \end{vmatrix} = 2\times(3+12) = 30$$

5.
$$\begin{vmatrix} 1 & -3 & 0 \\ -2 & 4 & 1 \\ 5 & -2 & 2 \end{vmatrix} \overset{R_2\to R_2+2R_1,\ R_3\to R_3-5R_1}{=} \begin{vmatrix} 1 & -3 & 0 \\ 0 & -2 & 1 \\ 0 & 13 & 2 \end{vmatrix} \overset{R_3\to R_3+\frac{13}{2}R_2}{=} \begin{vmatrix} 1 & -3 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & \frac{17}{2} \end{vmatrix} = -17$$

6.
$$\begin{vmatrix} 1 & 3 & 1 & 5 & 3 \\ -2 & -7 & 0 & -4 & 2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{vmatrix} \overset{R_2\to R_2+2R_1}{=} \begin{vmatrix} 1 & 3 & 1 & 5 & 3 \\ 0 & -1 & 2 & 6 & 8 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{vmatrix} \overset{R_4\to R_4-2R_3}{=} \begin{vmatrix} 1 & 3 & 1 & 5 & 3 \\ 0 & -1 & 2 & 6 & 8 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{vmatrix}$$
$$\overset{R_5\to R_5-R_4}{=} \begin{vmatrix} 1 & 3 & 1 & 5 & 3 \\ 0 & -1 & 2 & 6 & 8 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 2 \end{vmatrix} = -2$$

7.
$$\begin{vmatrix} 1 & -2 & 3 & 1 \\ 5 & -9 & 6 & 3 \\ -1 & 2 & -6 & -2 \\ 2 & 8 & 6 & 1 \end{vmatrix} \overset{R_2\to R_2-5R_1,\ R_3\to R_3+R_1,\ R_4\to R_4-2R_1}{=} \begin{vmatrix} 1 & -2 & 3 & 1 \\ 0 & 1 & -9 & -2 \\ 0 & 0 & -3 & -1 \\ 0 & 12 & 0 & -1 \end{vmatrix} \overset{R_4\to R_4-12R_2}{=} \begin{vmatrix} 1 & -2 & 3 & 1 \\ 0 & 1 & -9 & -2 \\ 0 & 0 & -3 & -1 \\ 0 & 0 & 108 & 23 \end{vmatrix}$$
$$\overset{R_4\to R_4+36R_3}{=} \begin{vmatrix} 1 & -2 & 3 & 1 \\ 0 & 1 & -9 & -2 \\ 0 & 0 & -3 & -1 \\ 0 & 0 & 0 & -13 \end{vmatrix} = 39$$

8.
$$\begin{vmatrix} -8 & 4 & -3 & 2 \\ 2 & 1 & -1 & -1 \\ -3 & -5 & 4 & 0 \\ 2 & -4 & 3 & -1 \end{vmatrix} \overset{R_1\to R_1+4R_2,\ R_4\to R_4-R_2,\ R_3\to R_3+R_2}{=} \begin{vmatrix} 0 & 8 & -7 & -2 \\ 2 & 1 & -1 & -1 \\ -1 & -4 & 3 & -1 \\ 0 & -5 & 4 & 0 \end{vmatrix} \overset{R_2\to R_2+2R_3}{=}$$
$$\begin{vmatrix} 0 & 8 & -7 & -2 \\ 0 & -7 & 5 & -3 \\ -1 & -4 & 3 & -1 \\ 0 & -5 & 4 & 0 \end{vmatrix} \overset{R_1\to R_1+R_2}{=} \begin{vmatrix} 0 & 1 & -2 & -5 \\ 0 & -7 & 5 & -3 \\ -1 & -4 & 3 & -1 \\ 0 & -5 & 4 & 0 \end{vmatrix} \overset{R_2\to R_2+7R_1,\ R_4\to R_4+5R_1}{=} \begin{vmatrix} 0 & 1 & -2 & -5 \\ 0 & 0 & -9 & -38 \\ -1 & -4 & 3 & -1 \\ 0 & 0 & -6 & -25 \end{vmatrix}$$
$$\overset{R_2\to R_1,\ R_3\to R_2,\ R_1\to R_3}{=} \begin{vmatrix} -1 & -4 & 3 & -1 \\ 0 & 1 & -2 & -5 \\ 0 & 0 & -9 & -38 \\ 0 & 0 & -6 & -25 \end{vmatrix} = -\begin{vmatrix} 1 & -2 & -5 \\ 0 & -9 & -38 \\ 0 & -6 & -25 \end{vmatrix} = -\begin{vmatrix} -9 & -38 \\ -6 & -25 \end{vmatrix} = 3$$

9.

$$\begin{vmatrix}1&-1&5&1\\-2&1&-7&1\\-3&2&-12&-2\\2&-1&9&1\end{vmatrix} \overset{R_2\to R_2+2R_1,\ R_3\to R_3+3R_1,\ R_4\to R_4-2R_1}{=} \begin{vmatrix}1&-1&5&1\\0&-1&3&3\\0&-1&3&1\\0&1&-1&-1\end{vmatrix} \overset{R_3\to R_3-R_2,\ R_4\to R_4+R_2}{=}$$

$$\begin{vmatrix}1&-1&5&1\\0&-1&3&3\\0&0&0&-2\\0&0&2&2\end{vmatrix} \overset{R_3\leftrightarrow R_4}{=} -\begin{vmatrix}1&-1&5&1\\0&-1&3&3\\0&0&2&2\\0&0&0&-2\end{vmatrix} = -4$$

10.

$$\begin{vmatrix}5&3&-8&4\\ \frac{15}{2}&\frac{1}{2}&-1&-7\\ -\frac{5}{2}&\frac{3}{2}&-4&1\\ 10&-3&8&-8\end{vmatrix} \overset{R_2\to R_2+3R_3,\ R_4\to R_4-2R_1}{=} \begin{vmatrix}5&3&-8&4\\0&5&-13&-4\\ -\frac{5}{2}&\frac{3}{2}&-4&1\\0&-9&24&-16\end{vmatrix} \overset{R_3\to R_3+\frac{1}{2}R_1}{=} \begin{vmatrix}5&3&-8&4\\0&5&-13&-4\\0&3&-8&3\\0&-9&24&-16\end{vmatrix}$$

$$\overset{R_4\to R_4+3R_3}{=} \begin{vmatrix}5&3&-8&4\\0&5&-13&-4\\0&3&-8&3\\0&0&0&-7\end{vmatrix} \overset{R_3\to R_3-\frac{3}{5}R_2}{=} \begin{vmatrix}5&3&-8&4\\0&5&-13&-4\\0&0&-\frac{1}{5}&\frac{27}{5}\\0&0&0&-7\end{vmatrix} = 35$$

11.

$$\begin{vmatrix}1&2&-1&3&0\\2&4&-3&1&-4\\2&6&4&8&-4\\-3&-8&-1&1&0\\1&3&3&10&1\end{vmatrix} \overset{R_2\to R_2-2R_1,\ R_3\to R_3-2R_1,\ R_4\to R_4+3R_1,\ R_5\to R_5-R_1}{=} \begin{vmatrix}1&2&-1&3&0\\0&0&-1&-5&-4\\0&2&6&2&-4\\0&-2&-4&10&0\\0&1&4&7&1\end{vmatrix}$$

$$\overset{R_3\to R_4+R_3,\ R_4\to R_4+2R_5}{=} \begin{vmatrix}1&2&-1&3&0\\0&0&-1&-5&-4\\0&0&2&12&-4\\0&0&4&24&2\\0&1&4&7&1\end{vmatrix} \overset{R_3\to R_3+2R_2,\ R_4\to R_4+4R_2}{=} \begin{vmatrix}1&2&-1&3&0\\0&0&-1&-5&-4\\0&0&0&2&-12\\0&0&0&4&-14\\0&1&4&7&1\end{vmatrix}$$

$$\overset{R_5\leftrightarrow R_2,\ R_4\to R_4-2R_3}{=} -\begin{vmatrix}1&2&-1&3&0\\0&1&4&7&1\\0&0&0&2&-12\\0&0&0&0&10\\0&0&-1&-5&-4\end{vmatrix} \overset{R_5\leftrightarrow R_3,\ R_4\leftrightarrow R_5}{=} -\begin{vmatrix}1&2&-1&3&0\\0&1&4&7&1\\0&0&-1&-5&-4\\0&0&0&2&-12\\0&0&0&0&10\end{vmatrix} = 20$$

12.

$$\begin{vmatrix}-4&1&1&1&1\\1&-4&1&1&1\\1&1&-4&1&1\\1&1&1&-4&1\\1&1&1&1&-4\end{vmatrix} \overset{R_5\to R_5-R_4,\ R_4\to R_4-R_3,\ R_3\to R_3-R_2,\ R_2\to R_2-R_1}{=} \begin{vmatrix}-4&1&1&1&1\\5&-5&0&0&0\\0&5&-5&0&0\\0&0&5&-5&0\\0&0&0&5&-5\end{vmatrix}$$

$$= 5^4 \begin{vmatrix} -4 & 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{vmatrix} \overset{R_1 \to R_1 + R5}{=} 5^4 \begin{vmatrix} -4 & 1 & 1 & 2 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{vmatrix} \overset{R_1 \to R_1 + 2R_4}{=}$$

$$5^4 \begin{vmatrix} -4 & 1 & 3 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{vmatrix} \overset{R_1 \to R_1 + 3R_3}{=} 5^4 \begin{vmatrix} -4 & 4 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{vmatrix} \overset{R_1 \to R_1 + 4R_2}{=}$$

$$5^4 \begin{vmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{vmatrix} = 0$$

**Question 24.** Prove that if a square matrix has two proportional rows or columns, then its determinant is zero.

*Solution.* Let $A$ be an $n \times n$ square matrix with two proportional rows, say $R_i = \alpha R_j$ for some scalar $\alpha$. Consider performing the row operation:

$$A \xrightarrow{R_i \to R_i - \alpha R_j} B.$$

Since this operation replaces $R_i$ with the zero row, the resulting matrix $B$ has a row of zeros. It is well known that a matrix with a zero row has a determinant of zero, so we conclude that

$$\det(A) = \det(B) = 0.$$

A similar argument holds if $A$ has two proportional columns.

**Question 25.** Compute the following determinants, given that

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = -6$$

1. $\begin{vmatrix} d & e & f \\ g & h & i \\ a & b & c \end{vmatrix}$

2. $\begin{vmatrix} g & h & i \\ d & e & f \\ a & b & c \end{vmatrix}$

3. $\begin{vmatrix} 3a & 3b & 3c \\ -d & -e & -f \\ 4g & 4h & 4i \end{vmatrix}$

4. $\begin{vmatrix} a+d & b+e & c+f \\ -d & -e & -f \\ g & h & i \end{vmatrix}$

5. $\begin{vmatrix} a+g & b+h & c+i \\ d & e & f \\ g & h & i \end{vmatrix}$

6. $\begin{vmatrix} a & b & c \\ d & e & f \\ 2a & 2b & 2c \end{vmatrix}$

7. $\begin{vmatrix} a & b & c \\ 2d & 2e & 2f \\ g+3a & h+3b & i+3c \end{vmatrix}$

8. $\begin{vmatrix} -3a & -3b & -3c \\ d & e & f \\ g-4d & h-4e & i-4f \end{vmatrix}$

*Solution.*

1.
$$\begin{vmatrix} d & e & f \\ g & h & i \\ a & b & c \end{vmatrix} \overset{R_3 \leftrightarrow R_1}{=} -\begin{vmatrix} a & b & c \\ g & h & i \\ d & e & f \end{vmatrix} \overset{R_2 \leftrightarrow R_3}{=} \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = -6$$

2.
$$\begin{vmatrix} g & h & i \\ d & e & f \\ a & b & c \end{vmatrix} \overset{R_1 \leftrightarrow R_3}{=} -\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 6$$

3.
$$\begin{vmatrix} 3a & 3b & 3c \\ -d & -e & -f \\ 4g & 4h & 4i \end{vmatrix} \overset{R_1 \to \frac{1}{3}R_1}{=} 3\begin{vmatrix} a & b & c \\ -d & -e & -f \\ 4g & 4h & 4i \end{vmatrix} \overset{R_2 \to -R_2}{=} -3\begin{vmatrix} a & b & c \\ d & e & f \\ 4g & 4h & 4i \end{vmatrix} \overset{R_3 \to \frac{1}{4}R_3}{=} -12\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 72$$

4.
$$\begin{vmatrix} a+d & b+e & c+f \\ -d & -e & -f \\ g & h & i \end{vmatrix} = \begin{vmatrix} a & b & c \\ -d & -e & -f \\ g & h & i \end{vmatrix} + \begin{vmatrix} d & e & f \\ -d & -e & -f \\ g & h & i \end{vmatrix} = -\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} + 0 = 6$$

5.
$$\begin{vmatrix} a+g & b+h & c+i \\ d & e & f \\ g & h & i \end{vmatrix} = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} + \begin{vmatrix} g & h & i \\ d & e & f \\ g & h & i \end{vmatrix} = -6 + 0 = -6$$

6.
$$\begin{vmatrix} a & b & c \\ d & e & f \\ 2a & 2b & 2c \end{vmatrix} = 0$$

7.
$$\begin{vmatrix} a & b & c \\ 2d & 2e & 2f \\ g+3a & h+3b & i+3c \end{vmatrix} = \begin{vmatrix} a & b & c \\ 2d & 2e & 2f \\ g & h & i \end{vmatrix} + \begin{vmatrix} a & b & c \\ 2d & 2e & 2f \\ 3a & 3b & 3c \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} + 0 = -12$$

8.
$$\begin{vmatrix} -3a & -3b & -3c \\ d & e & f \\ g-4d & h-4e & i-4f \end{vmatrix} = \begin{vmatrix} -3a & -3b & -3c \\ d & e & f \\ g & h & i \end{vmatrix} + \begin{vmatrix} -3a & -3b & -3c \\ d & e & f \\ -4d & -4e & -4f \end{vmatrix} = -3\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 18$$

**Question 26.** Use row reduction to show that

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (b-a)(c-a)(c-b)$$

*Solution.*

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} \overset{R_2\to R_2-aR_1,\ R_3\to R_3-a^2R_1}{=} \begin{vmatrix} 1 & 1 & 1 \\ 0 & b-a & c-a \\ 0 & b^2-a^2 & c^2-a^2 \end{vmatrix} \overset{R_3\to R_3-(a+b)R_2}{=}$$

$$\begin{vmatrix} 1 & 1 & 1 \\ 0 & b-a & c-a \\ 0 & 0 & (c^2-a^2)-(c-a)(a+b) \end{vmatrix} = (b-a)((c^2-a^2)-(c-a)(a+b))$$

$$= (b-a) = (b-a)(c-a)(c+a-a-b) = (b-a)(c-a)(c-b)$$

**Question 27.** Verify the following two formulas and make a conjecture about a general result of which these results are special cases

1. $\begin{vmatrix} 0 & 0 & a_{13} \\ 0 & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = -a_{13}a_{22}a_{31}$

2. $\begin{vmatrix} 0 & 0 & 0 & a_{14} \\ 0 & 0 & a_{23} & a_{24} \\ 0 & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} = a_{14}a_{23}a_{32}a_{41}$

*Solution.*

1.

$$\begin{vmatrix} 0 & 0 & a_{13} \\ 0 & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \overset{R_1\leftrightarrow R_3}{=} -\begin{vmatrix} a_{31} & a_{32} & a_{33} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{13} \end{vmatrix} = -a_{13}a_{22}a_{31}$$

2.

$$\begin{vmatrix} 0 & 0 & 0 & a_{14} \\ 0 & 0 & a_{23} & a_{24} \\ 0 & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} \overset{R_1\leftrightarrow R_4}{=} -\begin{vmatrix} a_{41} & a_{42} & a_{43} & a_{44} \\ 0 & 0 & a_{23} & a_{24} \\ 0 & a_{32} & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{14} \end{vmatrix} \overset{R_2\leftrightarrow R_3}{=} \begin{vmatrix} a_{41} & a_{42} & a_{43} & a_{44} \\ 0 & a_{32} & a_{33} & a_{34} \\ 0 & 0 & a_{23} & a_{24} \\ 0 & 0 & 0 & a_{14} \end{vmatrix} = a_{14}a_{23}a_{32}a_{41}$$

In general,

$$\begin{vmatrix} 0 & 0 & \cdots & 0 & a_{1n} \\ 0 & 0 & \cdots & a_{2(n-1)} & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{n(n-1)} & a_{nn} \end{vmatrix} = (-1)^{\ell} a_{n1}a_{(n-1)2}\cdots a_{2(n-1)}a_{1n},$$

where

$$\ell = \left\lfloor \frac{n}{2} \right\rfloor.$$

**Question 28.** Confirm the following identities

1. $\begin{vmatrix} a_1 & b_1 & a_1+b_1+c_1 \\ a_2 & b_2 & a_2+b_2+c_2 \\ a_3 & b_3 & a_3+b_3+c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

2. $\begin{vmatrix} a_1+b_1t & a_2+b_2t & a_3+b_3t \\ a_1t+b_1 & a_2t+b_2 & a_3t+b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = (1-t^2)\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

3. $\begin{vmatrix} a_1+b_1 & a_1-b_1 & c_1 \\ a_2+b_2 & a_2-b_2 & c_2 \\ a_3+b_3 & a_3-b_3 & c_3 \end{vmatrix} = -2\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

4. $\begin{vmatrix} a_1 & b_1+ta_1 & c_1+rb_1+sa_1 \\ a_2 & b_2+ta_2 & c_2+rb_2+sa_2 \\ a_3 & b_3+ta_3 & c_3+rb_3+sa_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

*Solution.*

1.

$$\begin{vmatrix} a_1 & b_1 & a_1+b_1+c_1 \\ a_2 & b_2 & a_2+b_2+c_2 \\ a_3 & b_3 & a_3+b_3+c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & a_1 \\ a_2 & b_2 & a_2 \\ a_3 & b_3 & a_3 \end{vmatrix} + \begin{vmatrix} a_1 & b_1 & b_1 \\ a_2 & b_2 & b_2 \\ a_3 & b_3 & b_3 \end{vmatrix} + \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

2.

$$\begin{aligned}
\begin{vmatrix} a_1+b_1t & a_2+b_2t & a_3+b_3t \\ a_1t+b_1 & a_2t+b_2 & a_3t+b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} &= \begin{vmatrix} a_1 & a_2 & a_3 \\ a_1t+b_1 & a_2t+b_2 & a_3t+b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} b_1t & b_2t & b_3t \\ a_1t+b_1 & a_2t+b_2 & a_3t+b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \\
&= \begin{vmatrix} a_1 & a_2 & a_3 \\ a_1t & a_2t & a_3t \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} b_1t & b_2t & b_3t \\ a_1t & a_2t & a_3t \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} b_1t & b_2t & b_3t \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \\
&= 0 + \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + t^2 \begin{vmatrix} b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + 0 = (1-t^2)\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}
\end{aligned}$$

3.

$$\begin{aligned}
\begin{vmatrix} a_1+b_1 & a_1-b_1 & c_1 \\ a_2+b_2 & a_2-b_2 & c_2 \\ a_3+b_3 & a_3-b_3 & c_3 \end{vmatrix} &= \begin{vmatrix} a_1 & a_1-b_1 & c_1 \\ a_2 & a_2-b_2 & c_2 \\ a_3 & a_3-b_3 & c_3 \end{vmatrix} + \begin{vmatrix} b_1 & a_1-b_1 & c_1 \\ b_2 & a_2-b_2 & c_2 \\ b_3 & a_3-b_3 & c_3 \end{vmatrix} \\
&= \begin{vmatrix} a_1 & a_1 & c_1 \\ a_2 & a_2 & c_2 \\ a_3 & a_3 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & -b_1 & c_1 \\ a_2 & -b_2 & c_2 \\ a_3 & -b_3 & c_3 \end{vmatrix} + \begin{vmatrix} b_1 & a_1 & c_1 \\ b_2 & a_2 & c_2 \\ b_3 & a_3 & c_3 \end{vmatrix} + \begin{vmatrix} b_1 & -b_1 & c_1 \\ b_2 & -b_2 & c_2 \\ b_3 & -b_3 & c_3 \end{vmatrix} \\
&= 0 - \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} - \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + 0 = -2\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}
\end{aligned}$$

4.

$$\begin{aligned}
\begin{vmatrix} a_1 & b_1+ta_1 & c_1+rb_1+sa_1 \\ a_2 & b_2+ta_2 & c_2+rb_2+sa_2 \\ a_3 & b_3+ta_3 & c_3+rb_3+sa_3 \end{vmatrix} &= \begin{vmatrix} a_1 & b_1 & c_1+rb_1+sa_1 \\ a_2 & b_2 & c_2+rb_2+sa_2 \\ a_3 & b_3 & c_3+rb_3+sa_3 \end{vmatrix} + \begin{vmatrix} a_1 & ta_1 & c_1+rb_1+sa_1 \\ a_2 & ta_2 & c_2+rb_2+sa_2 \\ a_3 & ta_3 & c_3+rb_3+sa_3 \end{vmatrix} \\
&= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & b_1 & rb_1 \\ a_2 & b_2 & rb_2 \\ a_3 & b_3 & rb_3 \end{vmatrix} + \begin{vmatrix} a_1 & b_1 & sa_1 \\ a_2 & b_2 & sa_2 \\ a_3 & b_3 & sa_3 \end{vmatrix} + 0 \\
&= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}
\end{aligned}$$

**Question 29.** For the following matrices, show that $\det(A) = 0$ without explicitly computing the determinant.

1. $A = \begin{pmatrix} -2 & 8 & 1 & 4 \\ 3 & 2 & 5 & 1 \\ 1 & 10 & 6 & 5 \\ 4 & -6 & 4 & -3 \end{pmatrix}$

2. $A = \begin{pmatrix} -4 & 1 & 1 & 1 & 1 \\ 1 & -4 & 1 & 1 & 1 \\ 1 & 1 & -4 & 1 & 1 \\ 1 & 1 & 1 & -4 & 1 \\ 1 & 1 & 1 & 1 & -4 \end{pmatrix}$

*Solution.*

1.
$$\det(A) \overset{R_1 \to R_1 + R_2 - R_3}{=} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 3 & 2 & 5 & 1 \\ 1 & 10 & 6 & 5 \\ 4 & -6 & 4 & -3 \end{pmatrix}$$

2.
$$\det(A) \overset{R_1 \to R_1 + R_2 + R_3 + R_4 + R_5}{=} \begin{vmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & -4 & 1 & 1 & 1 \\ 1 & 1 & -4 & 1 & 1 \\ 1 & 1 & 1 & -4 & 1 \\ 1 & 1 & 1 & 1 & -4 \end{vmatrix} = 0$$

**Question 30.** It can be proved that if a square matrix $M$ can be partitioned into *block triangular form* as

$$M = \begin{pmatrix} A & 0 \\ C & B \end{pmatrix} \quad \text{or} \quad M = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$$

where $A$ and $B$ are square matrices, then

$$\det(M) = \det(A)\det(B).$$

Use this result to compute the determinants of the following matrices:

1. $M = \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 8 & 6 & -9 \\ 2 & 5 & 0 & 4 & 7 & 5 \\ -1 & 3 & 2 & 6 & 9 & -2 \\ \hline 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & -3 & 8 & -4 \end{array}\right)$

2. $M = \left(\begin{array}{ccc|cc} 1 & 2 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 2 \\ 2 & 0 & 0 & 0 & 1 \end{array}\right)$

*Solution.*

1.
$$\det(M) = \begin{vmatrix} 1 & 2 & 0 \\ 2 & 5 & 0 \\ -1 & 3 & 2 \end{vmatrix} \begin{vmatrix} 3 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & 8 & -4 \end{vmatrix} = \left( \begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix} |2| \right) \left( \begin{vmatrix} 3 & 0 \\ 2 & 1 \end{vmatrix} |-4| \right) = 2 \times (-12) = -24$$

2.

$$\det(M) = \begin{vmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} = 1 \times 1 = 1$$

**Question 31.** Given $A \in \mathcal{M}_{n\times n}$. Let $B$ be the matrix that results when the rows of $A$ are written in reverse order. Formulate a statement that describes the relationship between $\det(A)$ and $\det(B)$.

*Solution.* Reversing the rows of $A$ can be seen as performing a sequence of row swaps. Specifically, we swap:

- The first row with the last row,
- The second row with the second-last row,
- And so on.

Thus, we can obtain $B$ from $A$ by repeatedly applying the row operation that swaps two rows. The number of such operations needed is

$$\left\lfloor \frac{n}{2} \right\rfloor.$$

We have

$$\det(B) = (-1)^{\lfloor \frac{n}{2} \rfloor} \det(A).$$

**Question 32.** Find the determinant of the following matrix.

$$\begin{pmatrix} a & b & b & b \\ b & a & b & b \\ b & b & a & b \\ b & b & b & a \end{pmatrix}$$

*Solution.*

$$\begin{vmatrix} a & b & b & b \\ b & a & b & b \\ b & b & a & b \\ b & b & b & a \end{vmatrix} \overset{R_1\to R_1-R_2,\ R_2\to R_2-R_3,\ R_2\to R_3-R_4}{=} \begin{vmatrix} a-b & b-a & 0 & 0 \\ 0 & a-b & b-a & 0 \\ 0 & 0 & a-b & b-a \\ b & b & b & a \end{vmatrix} = (a-b)^3 \begin{vmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ b & b & b & a \end{vmatrix}$$

$$\overset{R_1\to R_4-bR_1}{=} (a-b)^3 \begin{vmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 2b & b & a \end{vmatrix} = (a-b)^3 \begin{vmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 3b & a \end{vmatrix} = (a-b)^3 \begin{vmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & a+3b \end{vmatrix}$$

$$= (a-b)^3(a+3b)$$

**Question 33.** Determine whether the given matrix is invertible by calculating its determinant. If the matrix is invertible, compute its inverse using the adjugate method.

1. $A = \begin{pmatrix} 5 & 6 \\ -3 & -4 \end{pmatrix}$

2. $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

3. $A = \begin{pmatrix} -12 & 7 & -27 \\ 4 & -1 & 2 \\ 3 & 2 & -8 \end{pmatrix}$

4. $A = \begin{pmatrix} 31 & -20 & 106 \\ -11 & 7 & -37 \\ -9 & 6 & -32 \end{pmatrix}$

5. $A = \begin{pmatrix} 2 & 5 & 5 \\ -1 & -1 & 0 \\ 2 & 4 & 3 \end{pmatrix}$

6. $A = \begin{pmatrix} 4 & 2 & 8 \\ -2 & 1 & -4 \\ 3 & 1 & 6 \end{pmatrix}$

7. $A = \begin{pmatrix} 2 & 0 & 3 \\ 0 & 3 & 2 \\ -2 & 0 & -4 \end{pmatrix}$

8. $A = \begin{pmatrix} 1 & 0 & -1 \\ 9 & -1 & 4 \\ 8 & 9 & -1 \end{pmatrix}$

9. $A = \begin{pmatrix} 2 & -3 & 5 \\ 0 & 1 & -3 \\ 0 & 0 & 2 \end{pmatrix}$

10. $A = \begin{pmatrix} -3 & 0 & 1 \\ 5 & 0 & 6 \\ 8 & 0 & 3 \end{pmatrix}$

11. $A = \begin{pmatrix} 2 & 0 & 0 \\ 8 & 1 & 0 \\ -5 & 3 & 6 \end{pmatrix}$

12. $A = \begin{pmatrix} \sqrt{2} & -\sqrt{7} & 0 \\ 3\sqrt{2} & -3\sqrt{7} & 0 \\ 5 & -9 & 0 \end{pmatrix}$

13. $A = \begin{pmatrix} 2 & 0 & 3 \\ 0 & 3 & 2 \\ -1 & 0 & -4 \end{pmatrix}$

*Solution.*

1. $\det(A) = -2$

$$\operatorname{adj}(A) = \begin{pmatrix} -4 & -6 \\ 3 & 5 \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} 2 & 3 \\ -\frac{3}{2} & -\frac{5}{2} \end{pmatrix}$$

2. $\det(A) = 1$

$$\operatorname{adj}(A) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

3. $\det(A) = -79$

$$\operatorname{adj}(A) = \begin{pmatrix} 4 & 2 & -13 \\ 38 & 177 & -84 \\ 11 & 45 & -16 \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} -\frac{4}{79} & -\frac{2}{79} & \frac{13}{79} \\ -\frac{38}{79} & -\frac{177}{79} & \frac{84}{79} \\ -\frac{11}{79} & -\frac{45}{79} & \frac{16}{79} \end{pmatrix}$$

4. $\det(A) = 0$

5. $\det(A) = -1$

$$\operatorname{adj}(A) = \begin{pmatrix} -3 & 5 & 5 \\ 3 & -4 & -5 \\ -2 & 2 & 3 \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} 3 & -5 & -5 \\ -3 & 4 & 5 \\ 2 & -2 & -3 \end{pmatrix}$$

6. $\det(A) = 0$
7. $\det(A) = -6$

$$\operatorname{adj}(A) = \begin{pmatrix} -12 & 0 & -9 \\ -4 & -2 & -4 \\ 6 & 0 & 6 \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} 2 & 0 & \frac{3}{2} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -1 & 0 & -1 \end{pmatrix}$$

8. $\det(A) = -124$

$$\operatorname{adj}(A) = \begin{pmatrix} -35 & -9 & -1 \\ 41 & 7 & -13 \\ 89 & -9 & -1 \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} \frac{35}{124} & \frac{9}{124} & \frac{1}{124} \\ -\frac{41}{124} & -\frac{7}{124} & \frac{13}{124} \\ -\frac{89}{124} & \frac{9}{124} & \frac{1}{124} \end{pmatrix}$$

9. $\det(A) = 4$

$$\operatorname{adj}(A) = \begin{pmatrix} 2 & 6 & 4 \\ 0 & 4 & 6 \\ 0 & 0 & 2 \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{3}{2} & 1 \\ 0 & 1 & \frac{3}{2} \\ 0 & 0 & \frac{1}{2} \end{pmatrix}$$

10. $\det(A) = 0$
11. $\det(A) = 12$

$$\operatorname{adj}(A) = \begin{pmatrix} 6 & 0 & 0 \\ -48 & 12 & 0 \\ 29 & -6 & 2 \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ -4 & 1 & 0 \\ \frac{29}{12} & -\frac{1}{2} & \frac{1}{6} \end{pmatrix}$$

12. $\det(A) = 0$
13. $\det(A) = 15$

$$\operatorname{adj}(A) = \begin{pmatrix} -12 & 0 & -9 \\ -2 & -5 & -4 \\ 3 & 0 & 6 \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} \frac{4}{5} & 0 & \frac{3}{5} \\ \frac{2}{15} & \frac{1}{3} & \frac{4}{15} \\ -\frac{1}{5} & 0 & -\frac{2}{5} \end{pmatrix}$$

**Question 34.** Evaluate the determinant of the given matrix by

(a) cofactor expansion

(b) row operations

1. $A = \begin{pmatrix} 2 & -2 & -2 & -2 \\ -2 & 2 & 3 & 0 \\ -2 & -2 & 2 & 0 \\ 1 & -1 & -3 & -1 \end{pmatrix}$

2. $A = \begin{pmatrix} 1 & -1 & 0 & 0 \\ -3 & -3 & -1 & -1 \\ -1 & -1 & -3 & 2 \\ -1 & -2 & 2 & 1 \end{pmatrix}$

3. $A = \begin{pmatrix} -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & -1 & 1 & 1 & 0 \end{pmatrix}$

4. $A = \begin{pmatrix} 1 & 0 & -1 & 0 & -1 \\ -1 & -1 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 1 & 1 & 0 \\ -1 & 1 & 1 & -1 & 0 \end{pmatrix}$

*Solution.*

1. (a) Cofactor expansion along the fourth column

$$\det(A) = (-2) \times 12 + (-1) \times 8 = -32$$

(b)

$$\begin{aligned} |A| &= \begin{vmatrix} 2 & -2 & -2 & -2 \\ 0 & 0 & 1 & -2 \\ 0 & -4 & 0 & -2 \\ 1 & -1 & -3 & -1 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 4 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & -4 & 0 & -2 \\ 1 & -1 & -3 & -1 \end{vmatrix} = \begin{vmatrix} 1 & -1 & -3 & -1 \\ 0 & -4 & 0 & -2 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 4 & 0 \end{vmatrix} \\ &= \begin{vmatrix} 1 & -1 & -3 & -1 \\ 0 & -4 & 0 & -2 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 8 \end{vmatrix} = -32 \end{aligned}$$

2. (a) Cofactor expansion along the first row

$$\det(A) = 1 \times 32 + (-1) \times (-27) = 59$$

(b)

$$\begin{aligned} |A| &= \begin{vmatrix} 1 & -1 & 0 & 0 \\ 0 & -6 & -1 & -1 \\ 0 & -2 & -3 & 2 \\ 0 & -3 & 2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & -1 & 0 & 0 \\ 0 & -6 & -1 & -1 \\ 0 & 1 & -5 & 1 \\ 0 & -3 & 2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & -31 & 5 \\ 0 & 1 & -5 & 1 \\ 0 & 0 & -13 & 4 \end{vmatrix} = -\begin{vmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -5 & 1 \\ 0 & 0 & -31 & 5 \\ 0 & 0 & -13 & 4 \end{vmatrix} \\ &= -\begin{vmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -5 & 1 \\ 0 & 0 & -31 & 5 \\ 0 & 0 & 0 & \frac{59}{31} \end{vmatrix} = 59 \end{aligned}$$

3. (a) Cofactor expansion along the second row

$$\det(A) = (-1) \times 0 = 0$$

(b)

$$|A| = \begin{vmatrix} -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 2 & 1 & 0 \end{vmatrix} = -\begin{vmatrix} -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{vmatrix} = \begin{vmatrix} -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{vmatrix} = 0$$

4. (a) Cofactor expansion along the third row

$$\det(A) = 1 \times 4 + (-1) \times (-1) = 5$$

(b)

$$|A| = \begin{vmatrix} 1 & 0 & -1 & 0 & -1 \\ 0 & -1 & -1 & 0 & -2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & -1 & -1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & -1 & 0 & -1 \\ 0 & -1 & -1 & 0 & -2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & -1 & -1 & -3 \end{vmatrix} = \begin{vmatrix} 1 & 0 & -1 & 0 & -1 \\ 0 & -1 & -1 & 0 & -2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & -1 & -3 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & -1 & 0 & -1 \\ 0 & -1 & -1 & 0 & -2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & -5 \end{vmatrix} = 5$$

**Question 35.** Solve the following systems of equations using (if possible)

(a) the inverse of the coefficient matrix computed via the adjugate method

(b) Cramer's rule

1.

$$\begin{aligned} 7x_1 - 2x_2 &= 3 \\ 3x_1 + x_2 &= 5 \end{aligned}$$

2.

$$\begin{aligned} -9x - 4y &= 3 \\ -7x + 5y &= -10 \end{aligned}$$

3.

$$\begin{aligned} 2x + 3y &= 4 \\ 2x + 2y &= 4 \end{aligned}$$

4.

$$\begin{aligned} -10x - 7y &= -12 \\ 12x - 11y &= 5 \end{aligned}$$

5.

$$\begin{aligned} 5x - 5y &= 7 \\ 2x - 3y &= 6 \end{aligned}$$

6.

$$\begin{aligned} -x - 3y &= 4 \\ -8x + 4y &= 3 \end{aligned}$$

7.

$$\begin{aligned} 2x + 5y &= 4 \\ 4x + y &= 3 \end{aligned}$$

8.

$$\begin{aligned} x_1 + x_2 + 2x_3 &= 8 \\ -x_1 - 2x_2 + 3x_3 &= 1 \\ 3x_1 - 7x_2 + 4x_3 &= 10 \end{aligned}$$

9.

$$\begin{aligned} -2x + y - 4z &= -8 \\ -4y + z &= 3 \\ 4x - z &= -8 \end{aligned}$$

10.

$$\begin{aligned} 4x + 5y &= 2 \\ 11x + y + 2z &= 3 \\ x + 5y + 2z &= 1 \end{aligned}$$

11.

$$\begin{aligned} 2x + 3y + 2z &= -2 \\ -x - 3y - 8z &= -2 \\ -3x + 2y - 7z &= 2 \end{aligned}$$

12.

$$\begin{aligned} x - 4y + z &= 6 \\ 4x - y + 2z &= -1 \\ 2x + 2y - 3z &= -20 \end{aligned}$$

13.

$$\begin{aligned} x_1 - 3x_2 + x_3 &= 4 \\ 2x_1 - x_2 &= -2 \\ 4x_1 - 3x_3 &= 0 \end{aligned}$$

14.

$$\begin{aligned} 3x_1 - x_2 + x_3 &= 4 \\ -x_1 + 7x_2 - 3x_3 &= 1 \\ 2x_1 + 6x_2 - x_3 &= 5 \end{aligned}$$

15.

$$\begin{aligned} 2x_1 + 2x_2 + 2x_3 &= 0 \\ -2x_1 + 5x_2 + 2x_3 &= 1 \\ 8x_1 + x_2 + 4x_3 &= -1 \end{aligned}$$

16.

$$\begin{aligned} -x_1 - 4x_2 + 2x_3 + x_4 &= -32 \\ 2x_1 - x_2 + 7x_3 + 9x_4 &= 14 \\ -x_1 + x_2 + 3x_3 + x_4 &= 11 \\ x_1 - 2x_2 + x_3 - 4x_4 &= -4 \end{aligned}$$

17.

$$\begin{aligned} x - y + 2z - w &= -1 \\ 2x + y - 2z - 2w &= -2 \\ -x + 2y - 4z - 2w &= 1 \\ 3x - 3w &= -3 \end{aligned}$$

*Solution.*

1. (a) The coefficient matrix, its adjugate, and inverse are

$$A = \begin{pmatrix} 7 & -2 \\ 3 & 1 \end{pmatrix}, \quad \operatorname{adj}(A) = \begin{pmatrix} 1 & 2 \\ -3 & 7 \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} \frac{1}{13} & \frac{2}{13} \\ -\frac{3}{13} & \frac{7}{13} \end{pmatrix}$$

The solution of the linear system is $\begin{pmatrix} 1, & 2 \end{pmatrix}$.

(b)

$$\det(A) = 13, \quad \det(A_1) = 13, \quad \det(A_2) = 26$$

The solution for the linear system is given by

$$\left( \frac{13}{13}, \ \frac{26}{13} \right) = \begin{pmatrix} 1, & 2 \end{pmatrix}.$$

2. (a) The coefficient matrix, its adjugate, and inverse are

$$A = \begin{pmatrix} -9 & -4 \\ -7 & 5 \end{pmatrix}, \quad \operatorname{adj}(A) = \begin{pmatrix} 5 & 4 \\ 7 & -9 \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} -\frac{5}{73} & -\frac{4}{73} \\ -\frac{7}{73} & \frac{9}{73} \end{pmatrix}$$

The solution of the linear system is $\left( \frac{25}{73}, \ -\frac{111}{73} \right)$.

(b)

$$\det(A) = -73, \quad \det(A_1) = -25, \quad \det(A_2) = 111$$

The solution of the linear system is $\left( \frac{25}{73}, \ -\frac{111}{73} \right)$.

3. (a) The coefficient matrix, its adjugate, and inverse are

$$A = \begin{pmatrix} 2 & 3 \\ 2 & 2 \end{pmatrix}, \quad \operatorname{adj}(A) = \begin{pmatrix} 2 & -3 \\ -2 & 2 \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} -1 & \frac{3}{2} \\ 1 & -1 \end{pmatrix}$$

The solution of the linear system is $\begin{pmatrix} 2, & 0 \end{pmatrix}$.

(b)

$$\det(A) = -2, \quad \det(A_1) = -4, \quad \det(A_2) = 0$$

The solution of the linear system is $\begin{pmatrix} 2, & 0 \end{pmatrix}$.

4. (a) The coefficient matrix, its adjugate, and inverse are

$$A = \begin{pmatrix} -10 & -7 \\ 12 & -11 \end{pmatrix}, \quad \operatorname{adj}(A) = \begin{pmatrix} -11 & 7 \\ -12 & -10 \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} -\frac{11}{194} & \frac{7}{194} \\ -\frac{6}{97} & -\frac{5}{97} \end{pmatrix}$$

The solution of the linear system is $\left( \frac{167}{194}, \ \frac{47}{97} \right)$.

(b)

$$\det(A) = 194, \quad \det(A_1) = 167, \quad \det(A_2) = 94$$

The solution of the linear system is $\left(\frac{167}{194}, \frac{47}{97}\right)$.

5. (a) The coefficient matrix, its adjugate, and inverse are

$$A = \begin{pmatrix} 5 & -5 \\ 2 & -3 \end{pmatrix}, \quad \operatorname{adj}(A) = \begin{pmatrix} -3 & 5 \\ -2 & 5 \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} \frac{3}{5} & -1 \\ \frac{2}{5} & -1 \end{pmatrix}$$

The solution of the linear system is $\left(-\frac{9}{5}, -\frac{16}{5}\right)$.

(b)

$$\det(A) = -5, \quad \det(A_1) = 9, \quad \det(A_2) = 16.$$

The solution of the linear system is $\left(-\frac{9}{5}, -\frac{16}{5}\right)$.

6. (a) The coefficient matrix, its adjugate, and inverse are

$$A = \begin{pmatrix} -1 & -3 \\ -8 & 4 \end{pmatrix}, \quad \operatorname{adj}(A) = \begin{pmatrix} 4 & 3 \\ 8 & -1 \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} -\frac{1}{7} & -\frac{3}{28} \\ -\frac{2}{7} & \frac{1}{28} \end{pmatrix}$$

The solution of the linear system is $\left(-\frac{25}{28}, -\frac{29}{28}\right)$.

(b)

$$\det(A) = -28, \quad \det(A_1) = 25, \quad \det(A_2) = 29.$$

The solution of the linear system is $\left(-\frac{25}{28}, -\frac{29}{28}\right)$.

7. (a) The coefficient matrix, its adjugate, and inverse are

$$A = \begin{pmatrix} 2 & 5 \\ 4 & 1 \end{pmatrix}, \quad \operatorname{adj}(A) = \begin{pmatrix} 1 & -5 \\ -4 & 2 \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} -\frac{1}{18} & \frac{5}{18} \\ \frac{2}{9} & -\frac{1}{9} \end{pmatrix}.$$

(b)

$$\det(A) = -18, \quad \det(A_1) = -11, \quad \det(A_2) = -10$$

The solution of the linear system is $\left(\frac{11}{18}, \frac{5}{9}\right)$.

8. (a) The coefficient matrix, its adjugate, and inverse are

$$A = \begin{pmatrix} 1 & 1 & 2 \\ -1 & -2 & 3 \\ 3 & -7 & 4 \end{pmatrix}, \quad \operatorname{adj}(A) = \begin{pmatrix} 13 & -18 & 7 \\ 13 & -2 & -5 \\ 13 & 10 & -1 \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} \frac{1}{4} & -\frac{9}{26} & \frac{7}{52} \\ \frac{1}{4} & -\frac{1}{26} & -\frac{5}{52} \\ \frac{1}{4} & \frac{5}{26} & -\frac{1}{52} \end{pmatrix}.$$

(b)

$$\det(A) = 52, \quad \det(A_1) = 156, \quad \det(A_2) = 52, \quad \det(A_3) = 104.$$

The solution of the linear system is $\begin{pmatrix} 3, & 1, & 2 \end{pmatrix}$.

9. (a) The coefficient matrix, its adjugate, and inverse are

$$A = \begin{pmatrix} -2 & 1 & -4 \\ 0 & -4 & 1 \\ 4 & 0 & -1 \end{pmatrix}, \quad \operatorname{adj}(A) = \begin{pmatrix} 4 & 1 & -15 \\ 4 & 18 & 2 \\ 16 & 4 & 8 \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} -\frac{1}{17} & -\frac{1}{68} & \frac{15}{68} \\ -\frac{1}{17} & -\frac{9}{34} & -\frac{1}{34} \\ -\frac{4}{17} & -\frac{1}{17} & -\frac{2}{17} \end{pmatrix}.$$

(b)

$$\det(A) = -68, \quad \det(A_1) = 91, \quad \det(A_2) = 6, \quad \det(A_3) = -180$$

The solution of the linear system is $\left( -\frac{91}{68}, \ -\frac{3}{34}, \ \frac{45}{17} \right)$.

10. (a) The coefficient matrix, its adjugate, and inverse are

$$A = \begin{pmatrix} 4 & 5 & 0 \\ 11 & 1 & 2 \\ 1 & 5 & 2 \end{pmatrix}, \quad \operatorname{adj}(A) = \begin{pmatrix} -8 & -10 & 10 \\ -20 & 8 & -8 \\ 54 & -15 & -51 \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} \frac{2}{33} & \frac{5}{66} & -\frac{5}{66} \\ \frac{5}{33} & -\frac{2}{33} & \frac{2}{33} \\ -\frac{9}{22} & \frac{5}{44} & \frac{17}{44} \end{pmatrix}.$$

(b)

$$\det(A) = -132, \quad \det(A_1) = -36, \quad \det(A_2) = -24, \quad \det(A_3) = 12.$$

The solution of the linear system is $\left( \frac{3}{11}, \ \frac{2}{11}, \ -\frac{1}{11} \right)$.

11. (a) The coefficient matrix, its adjugate, and inverse are

$$A = \begin{pmatrix} 2 & 3 & 2 \\ -1 & -3 & -8 \\ -3 & 2 & -7 \end{pmatrix}, \quad \operatorname{adj}(A) = \begin{pmatrix} 37 & 25 & -18 \\ 17 & -8 & 14 \\ -11 & -13 & -3 \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} \frac{37}{103} & \frac{25}{103} & -\frac{18}{103} \\ \frac{17}{103} & -\frac{8}{103} & \frac{14}{103} \\ -\frac{11}{103} & -\frac{13}{103} & -\frac{3}{103} \end{pmatrix}.$$

(b)

$$\det(A) = 103, \quad \det(A_1) = -160, \quad \det(A_2) = 10, \quad \det(A_3) = 42.$$

The solution of the linear system is $\left( -\frac{160}{103}, \ \frac{10}{103}, \ \frac{42}{103} \right)$.

12. (a) The coefficient matrix, its adjugate, and inverse are

$$A = \begin{pmatrix} 1 & -4 & 1 \\ 4 & -1 & 2 \\ 2 & 2 & -3 \end{pmatrix}, \quad \operatorname{adj}(A) = \begin{pmatrix} -1 & -10 & -7 \\ 16 & -5 & 2 \\ 10 & -10 & 15 \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} \frac{1}{55} & \frac{2}{11} & \frac{7}{55} \\ -\frac{16}{55} & \frac{1}{11} & -\frac{2}{55} \\ -\frac{2}{11} & \frac{2}{11} & -\frac{3}{11} \end{pmatrix}.$$

(b)

$$\det(A) = -55, \quad \det(A_1) = 144, \quad \det(A_2) = 61, \quad \det(A_3) = -230,$$

The solution of the linear system is $\left(-\frac{144}{55}, -\frac{61}{55}, \frac{46}{11}\right)$.

13. (a) The coefficient matrix, its adjugate, and inverse are

$$A = \begin{pmatrix} 1 & -3 & 1 \\ 2 & -1 & 0 \\ 4 & 0 & -3 \end{pmatrix}, \quad \operatorname{adj}(A) = \begin{pmatrix} 3 & -9 & 1 \\ 6 & -7 & 2 \\ 4 & -12 & 5 \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} -\frac{3}{11} & \frac{9}{11} & -\frac{1}{11} \\ -\frac{6}{11} & \frac{7}{11} & -\frac{2}{11} \\ -\frac{4}{11} & \frac{12}{11} & -\frac{5}{11} \end{pmatrix}.$$

(b)

$$\det(A) = -11, \quad \det(A_1) = 30, \quad \det(A_2) = 38, \quad \det(A_3) = 40$$

The solution of the linear system is $\left(-\frac{30}{11}, -\frac{38}{11}, -\frac{40}{11}\right)$.

14. (a) The coefficient matrix, its adjugate, and inverse are

$$A = \begin{pmatrix} 3 & -1 & 1 \\ -1 & 7 & -3 \\ 2 & 6 & -1 \end{pmatrix}, \quad \operatorname{adj}(A) = \begin{pmatrix} 11 & 5 & -4 \\ -7 & -5 & 8 \\ -20 & -20 & 20 \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} \frac{11}{20} & \frac{1}{4} & -\frac{1}{5} \\ -\frac{7}{20} & -\frac{1}{4} & \frac{2}{5} \\ -1 & -1 & 1 \end{pmatrix}.$$

(b)

$$\det(A) = 20, \quad \det(A_1) = 29, \quad \det(A_2) = 7, \quad \det(A_3) = 0.$$

The solution of the linear system is $\left(\frac{29}{20}, \frac{7}{20}, 0\right)$.

15. The coefficient matrix is

$$A = \begin{pmatrix} 2 & 2 & 2 \\ -2 & 5 & 2 \\ 8 & 1 & 4 \end{pmatrix}, \quad \det(A) = 0.$$

16. (a) The coefficient matrix, its adjugate, and inverse are

$$A = \begin{pmatrix} -1 & -4 & 2 & 1 \\ 2 & -1 & 7 & 9 \\ -1 & 1 & 3 & 1 \\ 1 & -2 & 1 & -4 \end{pmatrix}, \quad \operatorname{adj}(A) = \begin{pmatrix} 90 & -63 & 117 & -90 \\ 83 & 3 & -66 & 11 \\ 8 & -15 & -93 & -55 \\ -17 & -21 & 39 & 64 \end{pmatrix},$$

$$A^{-1} = \begin{pmatrix} -\frac{10}{47} & \frac{7}{47} & -\frac{13}{47} & \frac{10}{47} \\ -\frac{83}{423} & -\frac{1}{141} & \frac{22}{141} & -\frac{11}{423} \\ -\frac{8}{423} & \frac{5}{141} & \frac{31}{141} & \frac{55}{423} \\ \frac{17}{423} & \frac{7}{141} & -\frac{13}{141} & -\frac{64}{423} \end{pmatrix}.$$

(b)

$$\det(A) = -423, \quad \det(A_1) = -2115, \quad \det(A_2) = -3384,$$
$$\det(A_3) = -1269, \quad \det(A_4) = 423.$$

The solution of the linear system is $\begin{pmatrix} 5, & 8, & 3, & -1 \end{pmatrix}$.

17. The coefficient matrix is

$$A = \begin{pmatrix} 1 & -1 & 2 & -1 \\ 2 & 1 & -2 & -2 \\ -1 & 2 & -4 & -2 \\ 3 & 0 & 0 & -3 \end{pmatrix}, \quad \det(A) = 0.$$