Tutorial 2

Vectors and Matrices

1 Vectors

Question 1. Given points A, B, C, find coordinates of B such that

a $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are orthogonal

b $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are parallel

1. $A = (3, 5),\ B = (2, y),\ C = (2, 8)$
2. $A = (-2, 5),\ B = (1, y),\ C = (4, -3)$
3. $A = (1, 5),\ B = (-1, y),\ C = (2, -3)$
4. $A = (2, 1),\ B = (x, -2),\ C = (1, 3)$

Solution.

1. We have
$$\boldsymbol{a} := \overrightarrow{AB} = \begin{bmatrix}-1, & y-5\end{bmatrix}, \quad \boldsymbol{b} := \overrightarrow{AC} = \begin{bmatrix}-1, & 3\end{bmatrix}$$
   a. If $\boldsymbol{a}$ and $\boldsymbol{b}$ are orthogonal, then
$$\boldsymbol{a} \cdot \boldsymbol{b} = 1 + 3y - 15 = 0 \Longrightarrow y = \frac{14}{3}$$
   b. If $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel, then $\exists \alpha \in \mathbb{R}$, $\alpha \neq 0$ s.t.
$$\boldsymbol{a} = \alpha\boldsymbol{b} \Longrightarrow \begin{bmatrix}-1, & y-5\end{bmatrix} = \alpha\begin{bmatrix}-1, & 3\end{bmatrix} \Longrightarrow \begin{bmatrix}-1, & y-5\end{bmatrix} = \begin{bmatrix}-\alpha, & 3\alpha\end{bmatrix},$$
   which gives
$$\alpha = 1, \quad y - 5 = 3\alpha \Longrightarrow y = 8.$$
2. We have
$$\boldsymbol{a} := \overrightarrow{AB} = \begin{bmatrix}3, & y-5\end{bmatrix}, \quad \boldsymbol{b} := \overrightarrow{AC} = \begin{bmatrix}6, & -8\end{bmatrix}$$
   a. If $\boldsymbol{a}$ and $\boldsymbol{b}$ are orthogonal, then
$$\boldsymbol{a} \cdot \boldsymbol{b} = 18 - 8y + 40 = 0 \Longrightarrow y = \frac{40+18}{8} = \frac{29}{4}.$$
   b. If $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel, then $\exists \alpha \in \mathbb{R}$, $\alpha \neq 0$ s.t.
$$\boldsymbol{a} = \alpha\boldsymbol{b} \Longrightarrow \begin{bmatrix}3, & y-5\end{bmatrix} = \alpha\begin{bmatrix}6, & -8\end{bmatrix} \Longrightarrow \begin{bmatrix}3, & y-5\end{bmatrix} = \begin{bmatrix}6\alpha, & -8\alpha\end{bmatrix},$$
   which gives
$$\alpha = \frac{1}{2}, \quad y - 5 = -8\alpha \Longrightarrow y = -4 + 5 = 1.$$
3. We have
$$\boldsymbol{a} := \overrightarrow{AB} = \begin{bmatrix}-2, & y-5\end{bmatrix}, \quad \boldsymbol{b} := \overrightarrow{AC} = \begin{bmatrix}1, & -8\end{bmatrix}$$

a. If $\boldsymbol{a}$ and $\boldsymbol{b}$ are orthogonal, then

$$\boldsymbol{a} \cdot \boldsymbol{b} = -2 - 8y + 40 = 0 \Longrightarrow y = \frac{38}{8} = \frac{19}{4}.$$

b. If $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel, then $\exists \alpha \in \mathbb{R}$, $\alpha \neq 0$ s.t.

$$\boldsymbol{a} = \alpha \boldsymbol{b} \Longrightarrow \begin{bmatrix} -2, & y-5 \end{bmatrix} = \alpha \begin{bmatrix} 1, & -8 \end{bmatrix} \Longrightarrow \begin{bmatrix} -2, & y-5 \end{bmatrix} = \begin{bmatrix} \alpha, & -8\alpha \end{bmatrix},$$

which gives

$$\alpha = -2, \quad y - 5 = -8\alpha \Longrightarrow y = 16 + 5 = 21.$$

4. We have

$$\boldsymbol{a} := \overrightarrow{AB} = \begin{bmatrix} x-2, & -3 \end{bmatrix}, \quad \boldsymbol{b} := \overrightarrow{AC} = \begin{bmatrix} -1, & 2 \end{bmatrix}$$

a. If $\boldsymbol{a}$ and $\boldsymbol{b}$ are orthogonal, then

$$\boldsymbol{a} \cdot \boldsymbol{b} = 2 - x - 6 = 0 \Longrightarrow x = -4.$$

b. If $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel, then $\exists \alpha \in \mathbb{R}$, $\alpha \neq 0$ s.t.

$$\boldsymbol{a} = \alpha \boldsymbol{b} \Longrightarrow \begin{bmatrix} x-2, & -3 \end{bmatrix} = \alpha \begin{bmatrix} -1, & 2 \end{bmatrix} \Longrightarrow \begin{bmatrix} x-2, & -3 \end{bmatrix} = \begin{bmatrix} -\alpha, & 2\alpha \end{bmatrix},$$

which gives

$$\alpha = -\frac{3}{2}, \quad x - 2 = -\alpha \Longrightarrow x = \frac{7}{2}.$$

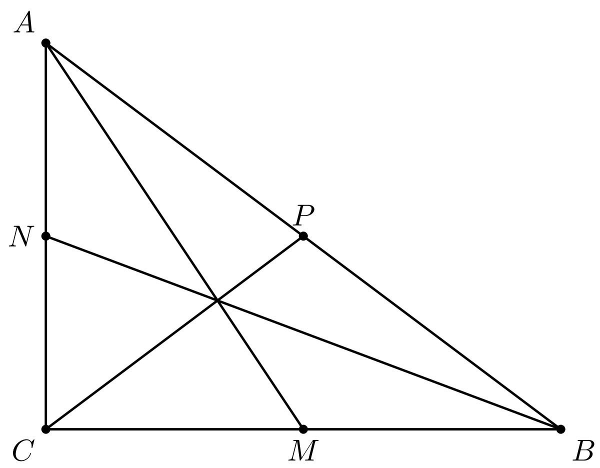
Question 2. Consider the triangle with vertices A, B, and C, $\triangle ABC$. Define the vectors

$$\boldsymbol{a} = \overrightarrow{BC}, \quad \boldsymbol{b} = \overrightarrow{AC}, \quad \boldsymbol{c} = \overrightarrow{AB}.$$

Let M, N, and P denote the midpoints of the sides BC, AC, and AB, respectively.

Determine the expressions for the vectors $\overrightarrow{AM}$, $\overrightarrow{BN}$, and $\overrightarrow{CP}$ in terms of $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$. Subsequently, rewrite these expressions using only the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$.

Solution.



$$\overrightarrow{AM} = \overrightarrow{AC} + \overrightarrow{CM} = \boldsymbol{b} - \frac{1}{2}\boldsymbol{a}, \quad \overrightarrow{BN} = \overrightarrow{BC} + \overrightarrow{CN} == \boldsymbol{a} - \frac{1}{2}\boldsymbol{b}, \quad \overrightarrow{CP} = \overrightarrow{CB} + \overrightarrow{BP} = -\boldsymbol{a} - \frac{1}{2}\boldsymbol{c}$$

Since

$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC},$$

we have

$$\boldsymbol{c} + \boldsymbol{a} = \boldsymbol{b} \Longrightarrow \boldsymbol{c} = \boldsymbol{b} - \boldsymbol{a},$$

which gives

$$\overrightarrow{CP} = -\boldsymbol{a} - \frac{1}{2}\boldsymbol{c} = -\boldsymbol{a} - \frac{1}{2}\boldsymbol{b} + \frac{1}{2}\boldsymbol{a} = -\frac{1}{2}\boldsymbol{a} - \frac{1}{2}\boldsymbol{b}$$

Question 3. Given points A, B, find x such that the norm of the vector $\overrightarrow{AB}$, $\|\overrightarrow{AB}\|$ equals the specified value d.

1. $A = (2, -3),\ B = (x, 0),\ d = 5$
2. $A = (1, 4),\ B = (x, 1),\ d = 4$
3. $A = (1, 5),\ B = (1, x),\ d = 6$

Solution.

1. By definition
$$\|\overrightarrow{AB}\| = \big\|\,[x-2, 3]\,\big\| = \sqrt{(x-2)^2 + 9}.$$
We have
$$(x-2)^2 + 9 = d^2 = 25 \Longrightarrow x^2 + 4 - 4x - 16 = 0 \Longrightarrow x^2 - 4x - 12 = 0 \Longrightarrow x = 6, -2$$
2. By definition
$$\|\overrightarrow{AB}\| = \big\|\,[x-1, -3]\,\big\| = \sqrt{(x-1)^2 + 9}.$$
We have
$$(x-1)^2 + 9 = d^2 = 16 \Longrightarrow x^2 + 1 - 2x - 7 \Longrightarrow x^2 - 2x - 6 = 0 \Longrightarrow x = 1 - \sqrt{7}, 1 + \sqrt{7}$$
3. By definition
$$\|\overrightarrow{AB}\| = \big\|\,[0, x-5]\,\big\| = \sqrt{(x-5)^2}.$$
We have
$$(x-5)^2 = d^2 = 6^2 \Longrightarrow x = 6 + 5, -6 + 5 \Longrightarrow x = 11, -1$$

Question 4. Prove that the triangle $\triangle ABC$ is isosceles.

1. $A = (-3, -2)$, $B = (1, 4)$, $C = (-5, 0)$.
2. $A = (7, -3, 6)$, $B = (11, -5, 3)$, $C = (10, -7, 8)$.

Solution.

1. Let
$$\boldsymbol{a} := \overrightarrow{AB} = \begin{bmatrix}4 & 6\end{bmatrix}, \quad \boldsymbol{b} := \overrightarrow{BC} = \begin{bmatrix}-6 & -4\end{bmatrix}, \quad \boldsymbol{c} := \overrightarrow{AC} = \begin{bmatrix}-2 & 2\end{bmatrix}.$$
And let $\theta_1, \theta_2, \theta_3$ denote the angles $\angle BAC, \angle ABC, \angle BCA$ respectively. Then
$$\begin{aligned}\cos\theta_1 &= \frac{\boldsymbol{a}\cdot\boldsymbol{c}}{\|\boldsymbol{a}\|\|\boldsymbol{c}\|} = \frac{-8+12}{\sqrt{16+36}\sqrt{4+4}} = \frac{4}{\sqrt{52}\sqrt{8}} = \frac{1}{\sqrt{26}}\\ \cos\theta_2 &= \frac{(-\boldsymbol{a})\cdot\boldsymbol{b}}{\|-\boldsymbol{a}\|\|\boldsymbol{b}\|} = \frac{24+24}{\sqrt{52}\sqrt{52}} = \frac{48}{52} = \frac{12}{13}\\ \cos\theta_3 &= \frac{(-\boldsymbol{b})\cdot(-\boldsymbol{c})}{\|-\boldsymbol{b}\|\|-\boldsymbol{c}\|} = \frac{12-8}{\sqrt{52}\sqrt{8}} = \frac{1}{\sqrt{26}}\end{aligned}$$
Since $\cos\theta_1 = \cos\theta_3$ and both angles lie in the range $(0,\pi)$, they must be equal.

2. Let
$$\boldsymbol{a} := \overrightarrow{AB} = \begin{bmatrix}4 & -2 & -3\end{bmatrix}, \quad \boldsymbol{b} := \overrightarrow{BC} = \begin{bmatrix}-1 & -2 & 5\end{bmatrix}, \quad \boldsymbol{c} := \overrightarrow{AC} = \begin{bmatrix}3 & -4 & 2\end{bmatrix}.$$
And let $\theta_1, \theta_2, \theta_3$ denote the angles $\angle BAC, \angle ABC, \angle BCA$ respectively. Then
$$\begin{aligned}\cos\theta_1 &= \frac{\boldsymbol{a}\cdot\boldsymbol{c}}{\|\boldsymbol{a}\|\|\boldsymbol{c}\|} = \frac{12+8-6}{\sqrt{16+4+9}\sqrt{9+16+4}} = \frac{14}{\sqrt{29}\sqrt{29}} = \frac{14}{29}\\ \cos\theta_2 &= \frac{(-\boldsymbol{a})\cdot\boldsymbol{b}}{\|-\boldsymbol{a}\|\|\boldsymbol{b}\|} = \frac{4-4+15}{\sqrt{15}\sqrt{1+4+25}} = \frac{15}{\sqrt{29}\sqrt{30}} = \frac{\sqrt{870}}{58}\\ \cos\theta_3 &= \frac{(-\boldsymbol{b})\cdot(-\boldsymbol{c})}{\|-\boldsymbol{b}\|\|-\boldsymbol{c}\|} = \frac{-3+8+10}{\sqrt{30}\sqrt{29}} = \frac{15}{\sqrt{29}\sqrt{30}} = \frac{\sqrt{870}}{58}\end{aligned}$$
Since $\cos\theta_2 = \cos\theta_3$ and both angles lie in the range $(0,\pi)$, they must be equal.

Question 5. Given vectors $\boldsymbol{a} = \begin{bmatrix}3, & -2\end{bmatrix}$ and $\boldsymbol{b} = \begin{bmatrix}-1, & 5\end{bmatrix}$, find vector $\boldsymbol{c}$ such that:

- $\boldsymbol{a}\cdot\boldsymbol{c} = 17$.
- $\boldsymbol{b}\cdot\boldsymbol{c} = 3$.

Solution. Let $\boldsymbol{c} = \begin{bmatrix}c_1, & c_2\end{bmatrix}$. Then by the definition of dot product, we have
$$\boldsymbol{a}\cdot\boldsymbol{c} = 3c_1 - 2c_2 = 17, \tag{1}$$
$$\boldsymbol{b}\cdot\boldsymbol{c} = -c_1 + 5c_2 = 3. \tag{2}$$
Next, multiply Equation (2) by 3 and add it to Equation (1) to eliminate c_1
$$(15-2)c_2 = 26 \Longrightarrow c_2 = \frac{26}{13} = 2.$$
Substituting $c_2 = 2$ into Equation (2), we find:
$$c_1 = 5c_2 - 3 = 10 - 3 = 7.$$
We have $\boldsymbol{c} = \begin{bmatrix}7, & 2\end{bmatrix}$

Question 6. Find vectors $\boldsymbol{a}$ such that it is orthogonal to the given vector $\boldsymbol{b}$ and a norm equal to the specified value d.

1. $\boldsymbol{b} = \begin{bmatrix} 3, & 4 \end{bmatrix}$, $d = 15$
2. $\boldsymbol{b} = \begin{bmatrix} -3, & 2 \end{bmatrix}$, $d = 10$
3. $\boldsymbol{b} = \begin{bmatrix} 1, & 4 \end{bmatrix}$, $d = 8$
4. $\boldsymbol{b} = \begin{bmatrix} -1, & 2 \end{bmatrix}$, $d = 7$
5. $\boldsymbol{b} = \begin{bmatrix} 2, & -5 \end{bmatrix}$, $d = 21$

Solution. Let $\boldsymbol{a} = \begin{bmatrix} a_1, & a_2 \end{bmatrix}$

1. By the given conditions, we have

$$\boldsymbol{a} \cdot \boldsymbol{b} = 3a_1 + 4a_2 = 0, \quad \|\boldsymbol{a}\| = \sqrt{a_1^2 + a_2^2} = 15.$$

Then we have the following two equations:

$$a_1 = -\frac{4}{3}a_2 \tag{3}$$
$$a_1^2 + a_2^2 = 15^2 \tag{4}$$

Substitute a_1 using Equation (3) in Equation (4) gives

$$a_2^2\left(\frac{16}{9}+1\right) = 225 \Longrightarrow a_2^2 = 225 \times \frac{9}{25} = 81 \Longrightarrow a_2 = \pm 9$$

Together with Equation (3) we have

$$a_1 = -\frac{4}{3} \times 9 = -12, \text{ or } a_1 = \frac{4}{3} \times 9 = 12.$$

Thus

$$\boldsymbol{a} = \begin{bmatrix} -12, & 9 \end{bmatrix} \text{ or } \begin{bmatrix} 12, & -9 \end{bmatrix}.$$

2. By the given conditions, we have

$$\boldsymbol{a} \cdot \boldsymbol{b} = -3a_1 + 2a_2 = 0, \quad \|\boldsymbol{a}\| = \sqrt{a_1^2 + a_2^2} = 10.$$

Then we have the following two equations:

$$a_1 = \frac{2}{3}a_2 \tag{5}$$
$$a_1^2 + a_2^2 = 100 \tag{6}$$

Substitute a_1 using Equation (5) in Equation (6) gives

$$a_2^2\left(\frac{4}{9}+1\right) = 100 \Longrightarrow a_2^2 = 100 \times \frac{9}{13} = \frac{900}{13} \Longrightarrow a_2 = \pm\frac{30\sqrt{13}}{13}$$

Together with Equation (5) we have

$$a_1 = \frac{2}{3} \times \frac{30\sqrt{13}}{13} = \frac{20\sqrt{13}}{13}, \text{ or } a_1 = -\frac{2}{3} \times \frac{30\sqrt{13}}{13} = -\frac{20\sqrt{13}}{13}.$$

Thus

$$\boldsymbol{a} = \begin{bmatrix} \frac{20\sqrt{13}}{13}, & \frac{30\sqrt{13}}{13} \end{bmatrix} \text{ or } \begin{bmatrix} -\frac{20\sqrt{13}}{13}, & -\frac{30\sqrt{13}}{13} \end{bmatrix}.$$

3. By the given conditions, we have

$$\boldsymbol{a}\cdot\boldsymbol{b} = a_1 + 4a_2 = 0, \quad \|\boldsymbol{a}\| = \sqrt{a_1^2 + a_2^2} = 8.$$

Then we have the following two equations:

$$\begin{aligned} a_1 &= -4a_2 & (7)\\ a_1^2 + a_2^2 &= 64 & (8)\end{aligned}$$

Substitute a_1 using Equation (7) in Equation (8) gives

$$a_2^2\,(16+1) = 64 \Longrightarrow a_2^2 = \frac{64}{17} \Longrightarrow a_2 = 8\frac{\sqrt{17}}{17}$$

Together with Equation (7) we have

$$a_1 = (-4)\times 8\frac{\sqrt{17}}{17} = -32\frac{\sqrt{17}}{17}, \text{ or } a_1 = 4\times 8\frac{\sqrt{17}}{17} = 32\frac{\sqrt{17}}{17}.$$

Thus

$$\boldsymbol{a} = \begin{bmatrix} 32\frac{\sqrt{17}}{17}, & -8\frac{\sqrt{17}}{17}\end{bmatrix} \text{ or } \begin{bmatrix} -32\frac{\sqrt{17}}{17}, & 8\frac{\sqrt{17}}{17}\end{bmatrix}.$$

4. By the given conditions, we have

$$\boldsymbol{a}\cdot\boldsymbol{b} = -a_1 + 2a_2 = 0, \quad \|\boldsymbol{a}\| = \sqrt{a_1^2 + a_2^2} = 7.$$

Then we have the following two equations:

$$\begin{aligned} a_1 &= 2a_2 & (9)\\ a_1^2 + a_2^2 &= 49 & (10)\end{aligned}$$

Substitute a_1 using Equation (9) in Equation (10) gives

$$a_2^2\,(4+1) = 49 \Longrightarrow a_2^2 = \frac{49}{5} \Longrightarrow a_2 = \pm 7\frac{\sqrt{5}}{5}$$

Together with Equation (9) we have

$$a_1 = 2\times 7\frac{\sqrt{5}}{5} = 14\frac{\sqrt{5}}{5}, \text{ or } a_1 = -2\times 7\frac{\sqrt{5}}{5} = -14\frac{\sqrt{5}}{5}.$$

Thus

$$\boldsymbol{a} = \begin{bmatrix} 14\frac{\sqrt{5}}{5}, & 7\frac{\sqrt{5}}{5}\end{bmatrix} \text{ or } \begin{bmatrix} -14\frac{\sqrt{5}}{5}, & -7\frac{\sqrt{5}}{5}\end{bmatrix}.$$

5. By the given conditions, we have

$$\boldsymbol{a}\cdot\boldsymbol{b} = 2a_1 - 5a_2 = 0, \quad \|\boldsymbol{a}\| = \sqrt{a_1^2 + a_2^2} = 21.$$

Then we have the following two equations:

$$\begin{aligned} a_1 &= \frac{5}{2}a_2 & (11)\\ a_1^2 + a_2^2 &= 441 & (12)\end{aligned}$$

Substitute a_1 using Equation (11) in Equation (12) gives

$$a_2^2\left(\frac{25}{4}+1\right) = 441 \Longrightarrow a_2^2 = \frac{441\times 4}{29} \Longrightarrow a_2 = \pm 42\frac{\sqrt{29}}{29}$$

Together with Equation (9) we have

$$a_1 = \frac{5}{2} \times 42\frac{\sqrt{29}}{29} = 105\frac{\sqrt{29}}{29}, \text{ or } a_1 = -\frac{5}{2} \times 42\frac{\sqrt{29}}{29} = -105\frac{\sqrt{29}}{29}.$$

Thus

$$\boldsymbol{a} = \begin{bmatrix} 105\frac{\sqrt{29}}{29}, & 42\frac{\sqrt{29}}{29} \end{bmatrix} \text{ or } \begin{bmatrix} -105\frac{\sqrt{29}}{29}, & -42\frac{\sqrt{29}}{29} \end{bmatrix}.$$

Question 7. Find the vector $\boldsymbol{c}$.

1. From the equation:

$$2\boldsymbol{c} + 3\boldsymbol{a} = \boldsymbol{b}, \quad \text{where} \quad \boldsymbol{a} = \begin{bmatrix} -1, & 2 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 0, & -2 \end{bmatrix}.$$

2. From the equation:

$$3\boldsymbol{c} - \boldsymbol{a} = 2\boldsymbol{b}, \quad \text{where} \quad \boldsymbol{a} = \begin{bmatrix} 3, & -1 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 0, & 2 \end{bmatrix}.$$

3. From the equation:

$$2\boldsymbol{c} - 2\boldsymbol{a} = \boldsymbol{b}, \quad \text{where} \quad \boldsymbol{a} = \begin{bmatrix} 3, & -1 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 0, & 2 \end{bmatrix}.$$

4. From the equation:

$$3\boldsymbol{c} + 5\boldsymbol{a} = 4\boldsymbol{b}, \quad \text{where} \quad \boldsymbol{a} = \begin{bmatrix} 2, & 4 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 1, & -2 \end{bmatrix}.$$

Solution. Let $\boldsymbol{c} = \begin{bmatrix} c_1, & c_2 \end{bmatrix}$.

1. By the given conditions, we have

$$\begin{aligned} 2\begin{bmatrix} c_1, & c_2 \end{bmatrix} + 3\begin{bmatrix} -1, & 2 \end{bmatrix} &= \begin{bmatrix} 0, & -2 \end{bmatrix} \\ \begin{bmatrix} 2c_1 - 3, & 2c_2 + 6 \end{bmatrix} &= \begin{bmatrix} 0, & -2 \end{bmatrix}, \end{aligned}$$

which gives

$$2c_1 - 3 = 0, \quad 2c_2 + 6 = -2.$$

And

$$c_1 = \frac{3}{2}, \quad c_2 = -4, \quad \boldsymbol{c} = \begin{bmatrix} \frac{3}{2}, & -4 \end{bmatrix}$$

2. By the given conditions, we have

$$\begin{aligned} 3\begin{bmatrix} c_1, & c_2 \end{bmatrix} - \begin{bmatrix} 3, & -1 \end{bmatrix} &= 2\begin{bmatrix} 0, & 2 \end{bmatrix} \\ \begin{bmatrix} 3c_1 - 3, & 3c_2 + 1 \end{bmatrix} &= \begin{bmatrix} 0, & 4 \end{bmatrix}, \end{aligned}$$

which gives

$$3c_1 - 3 = 0, \quad 3c_2 + 1 = 4.$$

And

$$c_1 = 1, \quad c_2 = 1, \quad \boldsymbol{c} = \begin{bmatrix} 1, & 1 \end{bmatrix}$$

3. By the given conditions, we have

$$\begin{aligned} 2\begin{bmatrix} c_1, & c_2 \end{bmatrix} - 2\begin{bmatrix} 3, & -1 \end{bmatrix} &= \begin{bmatrix} 0, & 2 \end{bmatrix} \\ \begin{bmatrix} 2c_1 - 6, & 2c_2 + 2 \end{bmatrix} &= \begin{bmatrix} 0, & 2 \end{bmatrix}, \end{aligned}$$

which gives

$$2c_1 - 6 = 0, \quad 2c_2 + 2 = 2.$$

And

$$c_1 = 3, \quad c_2 = 0, \quad \boldsymbol{c} = \begin{bmatrix} 3, & 0 \end{bmatrix}$$

4. By the given conditions, we have

$$\begin{aligned} 3\begin{bmatrix} c_1, & c_2 \end{bmatrix} + 5\begin{bmatrix} 2, & 4 \end{bmatrix} &= 4\begin{bmatrix} 1, & -2 \end{bmatrix} \\ \begin{bmatrix} 3c_1 + 10, & 3c_2 + 20 \end{bmatrix} &= \begin{bmatrix} 4, & -8 \end{bmatrix}, \end{aligned}$$

which gives

$$3c_1 + 10 = 4, \quad 3c_2 + 20 = -8.$$

And

$$c_1 = -2, \quad c_2 = -\frac{28}{3}, \quad \boldsymbol{c} = \begin{bmatrix} -2, & -\frac{28}{3} \end{bmatrix}$$

Question 8. Calculate the side lengths of the triangle $\triangle ABC$. Additionally, find the coordinates of a fourth point D such that the quadrilateral $ABCD$ forms a parallelogram.

1. $A = (-4, -2),\ B = (-1, 4), C = (2, 2)$
2. $A = (5, 1),\ B = (4, 2), C = (-1, 4)$
3. $A = (3, 2),\ B = (7, 4), C = (5, 6)$
4. $A = (-1, 2),\ B = (-3, 4), C = (-3, -3)$

Solution. Let $D = (d_1, d_2)$. Then $\overrightarrow{DC} = \overrightarrow{AB}$.

1. The side lengths of the triangle are given by

$$\begin{aligned} \|\overrightarrow{AB}\| &= \left\|\begin{bmatrix} -1+4, & 4+2 \end{bmatrix}\right\| = \left\|\begin{bmatrix} 3, & 6 \end{bmatrix}\right\| = \sqrt{9+36} = \sqrt{45} \\ \|\overrightarrow{BC}\| &= \left\|\begin{bmatrix} 2+1, & 2-4 \end{bmatrix}\right\| = \left\|\begin{bmatrix} 3, & -2 \end{bmatrix}\right\| = \sqrt{9+4} = \sqrt{13} \\ \|\overrightarrow{AC}\| &= \left\|\begin{bmatrix} 2+4, & 2+2 \end{bmatrix}\right\| = \left\|\begin{bmatrix} 6, & 4 \end{bmatrix}\right\| = \sqrt{36+16} = \sqrt{52} \end{aligned}$$

$\overrightarrow{DC} = \overrightarrow{AB}$ gives

$$\begin{bmatrix} 2 - d_1, & 2 - d_2 \end{bmatrix} = \begin{bmatrix} 3, & 6 \end{bmatrix} \Longrightarrow d_1 = 2 - 3 = -1,\ d_2 = 2 - 6 = -4.$$

We have $D = (-1, -4)$.

2. The side lengths of the triangle are given by

$$\begin{aligned} \|\overrightarrow{AB}\| &= \left\|\begin{bmatrix} 4-5, & 2-1 \end{bmatrix}\right\| = \left\|\begin{bmatrix} -1, & 1 \end{bmatrix}\right\| = \sqrt{1+1} = \sqrt{2} \\ \|\overrightarrow{BC}\| &= \left\|\begin{bmatrix} -1-4, & 4-2 \end{bmatrix}\right\| = \left\|\begin{bmatrix} -5, & 2 \end{bmatrix}\right\| = \sqrt{25+4} = \sqrt{29} \\ \|\overrightarrow{AC}\| &= \left\|\begin{bmatrix} -1-5, & 4-1 \end{bmatrix}\right\| = \left\|\begin{bmatrix} -6, & 3 \end{bmatrix}\right\| = \sqrt{36+9} = \sqrt{45} \end{aligned}$$

$\overrightarrow{DC} = \overrightarrow{AB}$ gives

$$\begin{bmatrix} -1 - d_1, & 4 - d_2 \end{bmatrix} = \begin{bmatrix} -1, & 1 \end{bmatrix} \Longrightarrow d_1 = 0,\ d_2 = 4 - 1 = 3.$$

We have $D = (0, 3)$.

3. The side lengths of the triangle are given by

$$\begin{aligned}
\|\overrightarrow{AB}\| &= \left\|\begin{bmatrix}7-3, & 4-2\end{bmatrix}\right\| = \left\|\begin{bmatrix}4, & 2\end{bmatrix}\right\| = \sqrt{16+4} = \sqrt{20} \\
\|\overrightarrow{BC}\| &= \left\|\begin{bmatrix}5-7, & 6-4\end{bmatrix}\right\| = \left\|\begin{bmatrix}-2, & 2\end{bmatrix}\right\| = \sqrt{4+4} = \sqrt{8} \\
\|\overrightarrow{AC}\| &= \left\|\begin{bmatrix}5-3, & 6-2\end{bmatrix}\right\| = \left\|\begin{bmatrix}2, & 4\end{bmatrix}\right\| = \sqrt{4+16} = \sqrt{20}
\end{aligned}$$

$\overrightarrow{DC} = \overrightarrow{AB}$ gives

$$\begin{bmatrix}5-d_1, & 6-d_2\end{bmatrix} = \begin{bmatrix}4, & 2\end{bmatrix} \Longrightarrow d_1 = 1,\ d_2 = 6-2 = 4.$$

We have $D = (1, 4)$.

4. The side lengths of the triangle are given by

$$\begin{aligned}
\|\overrightarrow{AB}\| &= \left\|\begin{bmatrix}-3+1, & 4-2\end{bmatrix}\right\| = \left\|\begin{bmatrix}-2, & 2\end{bmatrix}\right\| = \sqrt{4+4} = \sqrt{8} \\
\|\overrightarrow{BC}\| &= \left\|\begin{bmatrix}-3+3, & -3-4\end{bmatrix}\right\| = \left\|\begin{bmatrix}0, & 7\end{bmatrix}\right\| = \sqrt{0+49} = 7 \\
\|\overrightarrow{AC}\| &= \left\|\begin{bmatrix}-3+1, & -3-2\end{bmatrix}\right\| = \left\|\begin{bmatrix}-2, & -5\end{bmatrix}\right\| = \sqrt{4+25} = \sqrt{29}
\end{aligned}$$

$\overrightarrow{DC} = \overrightarrow{AB}$ gives

$$\begin{bmatrix}-3-d_1, & -3-d_2\end{bmatrix} = \begin{bmatrix}-2, & 2\end{bmatrix} \Longrightarrow d_1 = -1,\ d_2 = -5.$$

We have $D = (-1, -5)$.

Question 9. Calculate the lengths of the sides and diagonals of the specified quadrilateral $ABCD$, classify its type.

1. $A = (8, -4),\ B = (5, -6),\ C = (1, -4),\ D = (4, 2)$
2. $A = (-2, -3),\ B = (-5, -7),\ C = (-1, -10),\ D = (2, -6)$
3. $A = (7, 5),\ B = (-6, 2),\ C = (3, -1),\ D = (6, 1)$

Solution.

1. The length of the four sides are given by

$$\begin{aligned}
\|AB\| &= \left\|\begin{bmatrix}-3, & -2\end{bmatrix}\right\| = \sqrt{9+4} = \sqrt{13} \\
\|BC\| &= \left\|\begin{bmatrix}-4, & 2\end{bmatrix}\right\| = \sqrt{16+4} = \sqrt{20} \\
\|CD\| &= \left\|\begin{bmatrix}3, & 6\end{bmatrix}\right\| = \sqrt{9+36} = \sqrt{45} \\
\|DA\| &= \left\|\begin{bmatrix}4, & -6\end{bmatrix}\right\| = \sqrt{16+36} = \sqrt{52}
\end{aligned}$$

The length of the diagonals are given by

$$\|AC\| = \left\|\begin{bmatrix}-7, & 0\end{bmatrix}\right\| = 7, \quad \|BD\| = \left\|\begin{bmatrix}-1, & 8\end{bmatrix}\right\| = \sqrt{1+64} = \sqrt{65}$$

This quadrilateral is an irregular quadrilateral.

2. The length of the four sides are given by

$$\begin{aligned}\|AB\| &= \left\|\begin{bmatrix}-3, & -4\end{bmatrix}\right\| = \sqrt{9+16} = 5\\ \|BC\| &= \left\|\begin{bmatrix}4, & -3\end{bmatrix}\right\| = \sqrt{16+9} = 5\\ \|CD\| &= \left\|\begin{bmatrix}3, & 4\end{bmatrix}\right\| = \sqrt{9+16} = 5\\ \|DA\| &= \left\|\begin{bmatrix}-4, & 3\end{bmatrix}\right\| = \sqrt{16+36} = 5\end{aligned}$$

The length of the diagonals are given by

$$\|AC\| = \left\|\begin{bmatrix}-1, & -7\end{bmatrix}\right\| = \sqrt{50}, \quad \|BD\| = \left\|\begin{bmatrix}7, & 1\end{bmatrix}\right\| = \sqrt{50}$$

Furthermore, we have

$$\begin{aligned}\overrightarrow{BA}\cdot\overrightarrow{BC} &= \begin{bmatrix}3, & 4\end{bmatrix}\cdot\begin{bmatrix}4, & -3\end{bmatrix} = 0\\ \overrightarrow{AB}\cdot\overrightarrow{AD} &= \begin{bmatrix}-3, & -4\end{bmatrix}\cdot\begin{bmatrix}4, & -3\end{bmatrix} = 0\\ \overrightarrow{DA}\cdot\overrightarrow{DC} &= \begin{bmatrix}-4, & 3\end{bmatrix}\cdot\begin{bmatrix}-3, & -4\end{bmatrix} = 0\\ \overrightarrow{CD}\cdot\overrightarrow{CB} &= \begin{bmatrix}3, & 4\end{bmatrix}\cdot\begin{bmatrix}-4, & 3\end{bmatrix} = 0\end{aligned}$$

showing that the four angles are right angles. Thus this quadrilateral is a square.

3. The length of the four sides are given by

$$\begin{aligned}\|AB\| &= \left\|\begin{bmatrix}-13, & -3\end{bmatrix}\right\| = \sqrt{169+9} = \sqrt{178}\\ \|BC\| &= \left\|\begin{bmatrix}9, & -3\end{bmatrix}\right\| = \sqrt{81+9} = \sqrt{90}\\ \|CD\| &= \left\|\begin{bmatrix}3, & 2\end{bmatrix}\right\| = \sqrt{9+4} = \sqrt{13}\\ \|DA\| &= \left\|\begin{bmatrix}1, & 4\end{bmatrix}\right\| = \sqrt{1+16} = \sqrt{17}\end{aligned}$$

The length of the diagonals are given by

$$\|AC\| = \left\|\begin{bmatrix}-4, & -6\end{bmatrix}\right\| = \sqrt{16+36} = \sqrt{52}, \quad \|BD\| = \left\|\begin{bmatrix}12, & -1\end{bmatrix}\right\| = \sqrt{144+1} = \sqrt{145}$$

This quadrilateral is an irregular quadrilateral.

Question 10. Find the interior angles of the triangle $\triangle ABC$, where

$$A = (5\sqrt{3}, 5),\ B = (-\sqrt{3}, 1),\ C = (0, 0)$$

Solution. Let

$$\boldsymbol{a} := \overrightarrow{AB} = \begin{bmatrix}-6\sqrt{3}, & -4\end{bmatrix}, \quad \boldsymbol{b} := \overrightarrow{BC} = \begin{bmatrix}\sqrt{3}, & -1\end{bmatrix}, \quad \boldsymbol{c} := \overrightarrow{AC} = \begin{bmatrix}-5\sqrt{3}, & -5\end{bmatrix}.$$

And let$\theta_1, \theta_2, \theta_3$ denote the angles $\angle BAC, \angle ABC, \angle BCA$ respectively. Then

$$\begin{aligned}\cos\theta_1 &= \frac{\boldsymbol{a}\cdot\boldsymbol{c}}{\|\boldsymbol{a}\|\|\boldsymbol{c}\|} = \frac{90+20}{\sqrt{108+16}\sqrt{75+25}} = \frac{110}{\sqrt{124}\times 10} = \frac{11}{2\sqrt{31}} \Longrightarrow \theta_1 \approx 8.95^\circ\\ \cos\theta_2 &= \frac{(-\boldsymbol{a})\cdot\boldsymbol{b}}{\|-\boldsymbol{a}\|\|\boldsymbol{b}\|} = \frac{18-4}{\sqrt{124}\sqrt{3+1}} = \frac{7}{2\sqrt{31}} \Longrightarrow \theta_2 \approx 51.05^\circ\\ \cos\theta_3 &= \frac{(-\boldsymbol{b})\cdot(-\boldsymbol{c})}{\|-\boldsymbol{b}\|\|-\boldsymbol{c}\|} = \frac{-15+5}{2\times 10} = -\frac{1}{2} \Longrightarrow \theta_3 \approx 120^\circ\end{aligned}$$

Question 11. Identify which of the following vector pairs are parallel.

1. $\begin{bmatrix}-2, & 3, & 1\end{bmatrix}$, $\begin{bmatrix}6, & -4, & -3\end{bmatrix}$
2. $\begin{bmatrix}10, & -8, & 9, & 0, & 24\end{bmatrix}$, $\begin{bmatrix}\frac{5}{6}, & -\frac{2}{3}, & \frac{3}{4}, & 0, & 2\end{bmatrix}$
3. $\begin{bmatrix}8, & -6, & -9, & -8, & -2\end{bmatrix}$, $\begin{bmatrix}-\frac{32}{3}, & 8, & 12, & \frac{32}{3}, & \frac{8}{3}\end{bmatrix}$
4. $\begin{bmatrix}0, & 0, & 2, & 3, & 5\end{bmatrix}$, $\begin{bmatrix}0, & 1, & \frac{4}{3}, & 2, & \frac{10}{3}\end{bmatrix}$

Solution.

1. Suppose the two vectors were parallel, then there exists $\alpha \in \mathbb{R}$ such that
$$\begin{bmatrix}-2, & 3, & 1\end{bmatrix} = \alpha\begin{bmatrix}6, & -4, & -3\end{bmatrix}.$$
In particular, we have
$$-2 = 6\alpha \Longrightarrow \alpha = -\frac{1}{3}, \quad 3 = -4\alpha \Longrightarrow \alpha = -\frac{3}{4},$$
a contradiction. Therefore, the two vectors are not parallel.
2. Those two vectors are parallel because
$$\begin{bmatrix}10, & -8, & 9, & 0, & 24\end{bmatrix} = 12\begin{bmatrix}\frac{5}{6}, & -\frac{2}{3}, & \frac{3}{4}, & 0, & 2\end{bmatrix}$$
3. Those two vectors are parallel because
$$\begin{bmatrix}8, & -6, & -9, & -8, & -2\end{bmatrix} = -\frac{3}{4}\begin{bmatrix}-\frac{32}{3}, & 8, & 12, & \frac{32}{3}, & \frac{8}{3}\end{bmatrix}$$
4. Suppose the two vectors were parallel, then there exists $\alpha \in \mathbb{R}$ such that
$$\begin{bmatrix}0, & 0, & 2, & 3, & 5\end{bmatrix} = \alpha\begin{bmatrix}0, & 1, & \frac{4}{3}, & 2, & \frac{10}{3}\end{bmatrix}$$
In particular, we have
$$0 = \alpha \times 1,$$
a contradiction. Therefore, the two vectors are not parallel.

Question 12. Compute $\text{proj}_{\boldsymbol{a}}\boldsymbol{b}$, $\boldsymbol{b} - \text{proj}_{\boldsymbol{a}}\boldsymbol{b}$, $\text{proj}_{\boldsymbol{b}}\boldsymbol{a}$ and $\boldsymbol{a} - \text{proj}_{\boldsymbol{b}}\boldsymbol{a}$.

1. $\boldsymbol{a} = \begin{bmatrix}2, & 1, & 5\end{bmatrix}$, $\boldsymbol{b} = \begin{bmatrix}1, & 4, & -5\end{bmatrix}$
2. $\boldsymbol{a} = \begin{bmatrix}6, & 2, & 4\end{bmatrix}$, $\boldsymbol{b} = \begin{bmatrix}1, & -5, & 4\end{bmatrix}$
3. $\boldsymbol{a} = \begin{bmatrix}1, & 0, & 1, & -2\end{bmatrix}$, $\boldsymbol{b} = \begin{bmatrix}5, & -1, & 0, & 2\end{bmatrix}$
4. $\boldsymbol{a} = \begin{bmatrix}-2, & 3, & -4, & 1\end{bmatrix}$, $\boldsymbol{b} = \begin{bmatrix}5, & 1, & -8, & 2\end{bmatrix}$

Solution.

1. We have

$$\begin{aligned}
\mathrm{proj}_{\boldsymbol{a}}\boldsymbol{b} &= \frac{\boldsymbol{a}\cdot\boldsymbol{b}}{\|\boldsymbol{a}\|^2}\boldsymbol{a} = \frac{2+4-25}{4+1+25}\boldsymbol{a} = \frac{-19}{30}\begin{bmatrix}2, & 1, & 5\end{bmatrix} = \begin{bmatrix}-\frac{19}{15}, & -\frac{19}{30}, & -\frac{19}{6}\end{bmatrix},\\
\boldsymbol{b}-\mathrm{proj}_{\boldsymbol{a}}\boldsymbol{b} &= \begin{bmatrix}1+\frac{19}{15}, & 4+\frac{19}{30}, & -5+\frac{19}{6}\end{bmatrix} = \begin{bmatrix}\frac{34}{15}, & \frac{139}{30}, & -\frac{11}{6}\end{bmatrix}\\
\mathrm{proj}_{\boldsymbol{b}}\boldsymbol{a} &= \frac{\boldsymbol{a}\cdot\boldsymbol{b}}{\|\boldsymbol{b}\|^2}\boldsymbol{b} = \frac{-19}{1+16+25}\begin{bmatrix}1, & 4, & -5\end{bmatrix} = -\frac{19}{42}\begin{bmatrix}1, & 4, & -5\end{bmatrix} = \begin{bmatrix}-\frac{19}{42}, & -\frac{38}{21}, & \frac{95}{42}\end{bmatrix}\\
\boldsymbol{a}-\mathrm{proj}_{\boldsymbol{b}}\boldsymbol{a} &= \begin{bmatrix}2+\frac{19}{42}, & 1+\frac{38}{21}, & 5-\frac{95}{42}\end{bmatrix} = \begin{bmatrix}\frac{103}{42}, & \frac{59}{21}, & \frac{115}{42}\end{bmatrix}.
\end{aligned}$$

2. Answers:

$$\mathrm{proj}_{\boldsymbol{a}}\boldsymbol{b} = \begin{bmatrix}\frac{9}{7}, & \frac{3}{7}, & \frac{6}{7}\end{bmatrix}, \qquad \boldsymbol{b}-\mathrm{proj}_{\boldsymbol{a}}\boldsymbol{b} = \begin{bmatrix}-\frac{2}{7}, & -\frac{38}{7}, & \frac{22}{7}\end{bmatrix},$$
$$\mathrm{proj}_{\boldsymbol{b}}\boldsymbol{a} = \begin{bmatrix}\frac{2}{7}, & -\frac{10}{7}, & \frac{8}{7}\end{bmatrix}, \qquad \boldsymbol{a}-\mathrm{proj}_{\boldsymbol{b}}\boldsymbol{a} = \begin{bmatrix}\frac{40}{7}, & \frac{24}{7}, & \frac{20}{7}\end{bmatrix}$$

3. Answers:

$$\mathrm{proj}_{\boldsymbol{a}}\boldsymbol{b} = \begin{bmatrix}\frac{1}{6}, & 0, & \frac{1}{6}, & -\frac{1}{3}\end{bmatrix}, \qquad \boldsymbol{b}-\mathrm{proj}_{\boldsymbol{a}}\boldsymbol{b} = \begin{bmatrix}\frac{29}{6}, & -1, & -\frac{1}{6}, & \frac{7}{3}\end{bmatrix},$$
$$\mathrm{proj}_{\boldsymbol{b}}\boldsymbol{a} = \begin{bmatrix}\frac{1}{6}, & -\frac{1}{30}, & 0, & \frac{1}{15}\end{bmatrix}, \qquad \boldsymbol{a}-\mathrm{proj}_{\boldsymbol{b}}\boldsymbol{a} = \begin{bmatrix}\frac{5}{6}, & \frac{1}{30}, & 1, & -\frac{31}{15}\end{bmatrix}$$

4. Answers:

$$\mathrm{proj}_{\boldsymbol{a}}\boldsymbol{b} = \begin{bmatrix}-\frac{9}{5}, & \frac{27}{10}, & -\frac{18}{5}, & \frac{9}{10}\end{bmatrix}, \qquad \boldsymbol{b}-\mathrm{proj}_{\boldsymbol{a}}\boldsymbol{b} = \begin{bmatrix}\frac{34}{5}, & -\frac{17}{10}, & -\frac{22}{5}, & \frac{11}{10}\end{bmatrix},$$
$$\mathrm{proj}_{\boldsymbol{b}}\boldsymbol{a} = \begin{bmatrix}\frac{135}{94}, & \frac{27}{94}, & -\frac{108}{47}, & \frac{27}{47}\end{bmatrix}, \qquad \boldsymbol{a}-\mathrm{proj}_{\boldsymbol{b}}\boldsymbol{a} = \begin{bmatrix}-\frac{323}{94}, & \frac{255}{94}, & -\frac{80}{47}, & \frac{20}{47}\end{bmatrix}$$

Question 13. For any $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^n$, show that $\boldsymbol{b} - \mathrm{proj}_{\boldsymbol{a}}\boldsymbol{b}$ is orthogonal to $\boldsymbol{a}$.

Solution. Take any $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^n$, we have

$$\boldsymbol{a}\cdot(\boldsymbol{b}-\mathrm{proj}_{\boldsymbol{a}}\boldsymbol{b}) = \boldsymbol{a}\cdot\boldsymbol{b} - \boldsymbol{a}\cdot\frac{\boldsymbol{a}\cdot\boldsymbol{b}}{\|\boldsymbol{a}\|^2}\boldsymbol{a} = \boldsymbol{a}\cdot\boldsymbol{b} - \frac{\boldsymbol{a}\cdot\boldsymbol{b}}{\|\boldsymbol{a}\|^2}(\boldsymbol{a}\cdot\boldsymbol{a}) = \boldsymbol{a}\cdot\boldsymbol{b} - \boldsymbol{a}\cdot\boldsymbol{b} = 0.$$

Therefore, $\boldsymbol{b} - \mathrm{proj}_{\boldsymbol{a}}\boldsymbol{b}$ is orthogonal to $\boldsymbol{a}$.

Question 14. Take any $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^n$, where both vectors are nonzero.

1. Suppose $\boldsymbol{a}$ and $\boldsymbol{b}$ are orthogonal. Find $\mathrm{proj}_{\boldsymbol{a}}\boldsymbol{b}$.
2. Suppose $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel. Find $\mathrm{proj}_{\boldsymbol{a}}\boldsymbol{b}$.

Solution.

1. If $\boldsymbol{a}$ and $\boldsymbol{b}$ are orthogonal, by definition, we have $\boldsymbol{a}\cdot\boldsymbol{b} = 0$. Then

$$\mathrm{proj}_{\boldsymbol{a}}\boldsymbol{b} = \frac{\boldsymbol{a}\cdot\boldsymbol{b}}{\|\boldsymbol{a}\|^2}\boldsymbol{a} = \mathbf{0}.$$

2. If $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel, by definition, there exists $\alpha \in \mathbb{R}$, $\alpha \neq 0$ such that $\boldsymbol{a} = \alpha\boldsymbol{b}$. Then

$$\mathrm{proj}_{\boldsymbol{a}}\boldsymbol{b} = \frac{\boldsymbol{a}\cdot\boldsymbol{b}}{\|\boldsymbol{a}\|^2}\boldsymbol{a} = \frac{\alpha\boldsymbol{a}\cdot\boldsymbol{a}}{\|\boldsymbol{a}\|^2}\boldsymbol{a} = \alpha\boldsymbol{a} = \boldsymbol{b}.$$

2 Matrices

Question 1. Let

$$A = \begin{bmatrix} -4 & 2 & 3 \\ 0 & 5 & -1 \\ 6 & 1 & -2 \end{bmatrix} \quad B = \begin{bmatrix} 6 & -1 & 0 \\ 2 & 2 & -4 \\ 3 & -1 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 5 & -1 \\ -3 & 4 \end{bmatrix}$$

$$D = \begin{bmatrix} -7 & 1 & -4 \\ 3 & -2 & 8 \end{bmatrix} \quad E = \begin{bmatrix} 3 & -3 & 5 \\ 1 & 0 & -2 \\ 6 & 7 & -2 \end{bmatrix} \quad F = \begin{bmatrix} 8 & -1 \\ 2 & 0 \\ 5 & -3 \end{bmatrix}$$

Compute, if possible, the following matrices.

1. $A + B$
2. $2A - 3E - B$
3. $2C^\top - 3F$
4. $C + D$
5. $2D - 3F$
6. $5(F^\top - D^\top)$
7. $4A$
8. $A^\top + E^\top$
9. $\left((B - A)^\top + E^\top\right)^\top$
10. $2A - 3B$
11. $(A + E)^\top$
12. $A - B + E$
13. $C + 3F - E$
14. $4D + 3F^\top$

Solution.

1. $A + B = \begin{bmatrix} 2 & 1 & 3 \\ 2 & 7 & -5 \\ 9 & 0 & 3 \end{bmatrix}$
2. $2A - 3E - B = \begin{bmatrix} -23 & 14 & -9 \\ -5 & 8 & 8 \\ -9 & -18 & 9 \end{bmatrix}$
3. $2C^\top - 3F$ is not defined
4. $C + D$ is not defined
5. $2D - 3F$ is not defined
6. $5(F^\top - D^\top)$ is not defined
7. $4A = \begin{bmatrix} -16 & 8 & 12 \\ 0 & 20 & -4 \\ 24 & 4 & 8 \end{bmatrix}$
8. $A^\top + E^\top = \begin{bmatrix} -1 & 1 & 12 \\ -1 & 5 & 8 \\ 8 & -3 & 0 \end{bmatrix}$
9. $\left((B - A)^\top + E^\top\right)^\top = \begin{bmatrix} 13 & -6 & 2 \\ 3 & -3 & -5 \\ 3 & 5 & -3 \end{bmatrix}$
10. $2A - 3B = \begin{bmatrix} -26 & 7 & 6 \\ -6 & 4 & 10 \\ 3 & 5 & 1 \end{bmatrix}$
11. $(A + E)^\top = \begin{bmatrix} -1 & 1 & 12 \\ -1 & 5 & 8 \\ 8 & -3 & 0 \end{bmatrix}$
12. $A - B + E = \begin{bmatrix} -7 & 0 & 8 \\ -1 & 3 & 1 \\ 9 & 9 & -1 \end{bmatrix}$
13. $C + 3F - E$ is not defined
14. $4D + 3F^\top = \begin{bmatrix} -4 & 10 & -1 \\ 9 & -8 & 23 \end{bmatrix}$

For Questions 2-5, let

$$A = \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix}, \quad E = \begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix}$$

Question 2. Compute, if possible, the following matrices.

1. $D + E$
2. $-3(D + 2E)$
3. $D - E$
4. $A - A$
5. $5A$
6. $\mathrm{tr}\,(D)$
7. $-7D$
8. $\mathrm{tr}\,(D - 3E)$
9. $2B - C$
10. $4\mathrm{tr}\,(7B)$
11. $4E - 2D$
12. $\mathrm{tr}\,(A)$

Solution.

1. $D + E = \begin{bmatrix} 7 & 6 & 5 \\ -2 & 1 & 3 \\ 7 & 3 & 7 \end{bmatrix}$
2. $-3(D + 2E) = \begin{bmatrix} -39 & -21 & -24 \\ 9 & -6 & -15 \\ -33 & -12 & -30 \end{bmatrix}$
3. $D - E = \begin{bmatrix} -5 & 4 & -1 \\ 0 & -1 & -1 \\ -1 & 1 & 1 \end{bmatrix}$
4. $A - A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$
5. $5A = \begin{bmatrix} 15 & 0 \\ -5 & 10 \\ 5 & 5 \end{bmatrix}$
6. $\mathrm{tr}\,(D) = 5$
7. $-7D = \begin{bmatrix} -7 & -35 & -14 \\ 7 & 0 & -7 \\ -21 & -14 & -28 \end{bmatrix}$
8. $\mathrm{tr}\,(D - 3E) = -25$
9. $2B - C$ is not defined
10. $4\mathrm{tr}\,(7B) = 168$
11. $4E - 2D = \begin{bmatrix} 22 & -6 & 8 \\ -2 & 4 & 6 \\ 10 & 0 & 4 \end{bmatrix}$
12. $\mathrm{tr}\,(A) = 5$

Question 3. Compute, if possible, the following matrices.

1. $2A^\top + C$
2. $2E^\top - 3D^\top$
3. $D^\top - E^\top$
4. $(2E^\top - 3D^\top)^\top$
5. $(D - E)^\top$
6. $(CD)E$
7. $B^\top + 5C^\top$
8. $C(AB)$
9. $\frac{1}{2}C^\top - \frac{1}{4}A$
10. $\mathrm{tr}\left(DE^\top\right)$
11. $B - B^\top$
12. $\mathrm{tr}\,(BC)$

Solution.

1. $2A^\top + C = \begin{bmatrix} 7 & 2 & 4 \\ 3 & 5 & 7 \end{bmatrix}$
2. $2E^\top - 3D^\top = \begin{bmatrix} 9 & 1 & -1 \\ -13 & 2 & -4 \\ 0 & 1 & -6 \end{bmatrix}$

3. $D^\top - E^\top = \begin{bmatrix} -5 & 0 & -1 \\ 4 & -1 & 1 \\ -1 & -1 & 1 \end{bmatrix}$

4. $(2E^\top - 3D^\top)^\top = \begin{bmatrix} 9 & -13 & 0 \\ 1 & 2 & 1 \\ -1 & -4 & -6 \end{bmatrix}$

5. $(D - E)^\top = \begin{bmatrix} -5 & 0 & -1 \\ 4 & -1 & 1 \\ -1 & -1 & 1 \end{bmatrix}$

6. $(CD)E = \begin{bmatrix} 65 & 26 & 69 \\ 185 & 69 & 182 \end{bmatrix}$

7. $B^\top + 5C^\top$ is not defined

8. $C(AB) = \begin{bmatrix} 4 & 19 \\ 52 & 1 \end{bmatrix}$

9. $\frac{1}{2}C^\top - \frac{1}{4}A = \begin{bmatrix} -0.25 & 1.5 \\ 2.25 & 0.0 \\ 0.75 & 2.25 \end{bmatrix}$

10. $\operatorname{tr}\left(DE^\top\right) = 18$

11. $B - B^\top = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

12. $\operatorname{tr}(BC)$ is not defined, BC is not a square matrix

Question 4. Compute, if possible, the following matrices.

1. AB
2. BA
3. $(3E)D$
4. $(AB)C$
5. $A(BC)$
6. $CC^\top$
7. $(DA)^\top$
8. $(C^\top B)A^\top$
9. $\operatorname{tr}\left(DD^\top\right)$
10. $\operatorname{tr}\left(4E^\top - D\right)$
11. $\operatorname{tr}\left(C^\top A^\top + 2E^\top\right)$
12. $\operatorname{tr}\left((EC^\top)^\top A\right)$

Solution.

1. $AB = \begin{bmatrix} 12 & -3 \\ -4 & 5 \\ 4 & 1 \end{bmatrix}$

2. BA is not defined

3. $(3E)D = \begin{bmatrix} 42 & 108 & 75 \\ 12 & -3 & 21 \\ 36 & 78 & 63 \end{bmatrix}$

4. $(AB)C = \begin{bmatrix} 3 & 45 & 9 \\ 11 & -11 & 17 \\ 7 & 17 & 13 \end{bmatrix}$

5. $A(BC) = \begin{bmatrix} 3 & 45 & 9 \\ 11 & -11 & 17 \\ 7 & 17 & 13 \end{bmatrix}$

6. $CC^\top = \begin{bmatrix} 21 & 17 \\ 17 & 35 \end{bmatrix}$

7. $(DA)^\top = \begin{bmatrix} 0 & -2 & 11 \\ 12 & 1 & 8 \end{bmatrix}$

8. $(C^\top B)A^\top = \begin{bmatrix} 12 & 6 & 9 \\ 48 & -20 & 14 \\ 24 & 8 & 16 \end{bmatrix}$

9. $\operatorname{tr}\left(DD^\top\right) = 61$

10. $\operatorname{tr}\left(4E^\top - D\right) = 35$

11. $\operatorname{tr}\left(C^\top A^\top + 2E^\top\right) = 28$

12. $\operatorname{tr}\left((EC^\top)^\top A\right) = 99$

Question 5. Compute, if possible, the following matrices.

1. $(2D^\top - E)A$
2. $(BA^\top - 2C)^\top$
3. $(4B)C + 2B$
4. $B^\top(CC^\top - A^\top A)$
5. $(-AC)^\top + 5D^\top$
6. $D^\top E^\top - (ED)^\top$

Solution.

1. $(2D^\top - E)A = \begin{bmatrix} -6 & -3 \\ 36 & 0 \\ 4 & 7 \end{bmatrix}$
2. $(BA^\top - 2C)^\top = \begin{bmatrix} 10 & -6 \\ -14 & 2 \\ -1 & -8 \end{bmatrix}$
3. $(4B)C + 2B$ is not defined
4. $B^\top(CC^\top - A^\top A)$ is not defined
5. $(-AC)^\top + 5D^\top = \begin{bmatrix} 2 & -10 & 11 \\ 13 & 2 & 5 \\ 4 & -3 & 13 \end{bmatrix}$
6. $D^\top E^\top - (ED)^\top = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Question 6. Given matrices

$$A \in \mathcal{M}_{4\times5}, \quad B \in \mathcal{M}_{4\times5}, \quad C \in \mathcal{M}_{5\times2}, \quad D \in \mathcal{M}_{4\times2}, \quad E \in \mathcal{M}_{5\times4},$$

for each of the cases below, determine whether the given computation is defined. If it is, specify the size of the resulting matrix.

1. BA
2. $A - 3E^\top$
3. $BC - 3D$
4. $AB^\top$
5. $E(5B + A)$
6. $D^\top(BE)$
7. $AC + D$
8. $CD^\top$
9. $B^\top D + ED$
10. $E(AC)$
11. DC
12. $BA^\top + D$

Solution.

1. BA is not defined
2. $A - 3E^\top \in \mathcal{M}_{4\times5}$
3. $BC - 3D \in \mathcal{M}_{4\times2}$
4. $AB^\top \in \mathcal{M}_{4\times4}$
5. $E(5B + A) \in \mathcal{M}_{5\times5}$
6. $D^\top(BE) \in \mathcal{M}_{2\times4}$
7. $AC + D \in \mathcal{M}_{4\times2}$
8. $CD^\top \in \mathcal{M}_{5\times4}$
9. $B^\top D + ED \in \mathcal{M}_{5\times2}$
10. $E(AC) \in \mathcal{M}_{5\times2}$
11. DC is not defined
12. $BA^\top + D$ is not defined

Question 7. Determine which of the following matrices are square, diagonal, upper or lower triangular, symmetric, or skew-symmetric. Compute the transpose of each matrix.

$$A = \begin{bmatrix} -1 & 4 \\ 0 & 1 \\ 6 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}, \quad C = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} -1 \\ 4 \\ 2 \end{bmatrix}$$

$$E = \begin{bmatrix} 0 & 0 & 6 \\ 0 & -6 & 0 \\ -6 & 0 & 0 \end{bmatrix} \qquad F = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} \qquad G = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

$$H = \begin{bmatrix} 0 & -1 & 6 & 2 \\ 1 & 0 & -7 & 1 \\ -6 & 7 & 0 & -4 \\ -2 & -1 & 4 & 0 \end{bmatrix} \qquad J = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \qquad K = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -2 & 1 & 5 & 6 \\ -3 & -5 & 1 & 7 \\ -4 & -6 & -7 & 1 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad M = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} \qquad N = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad Q = \begin{bmatrix} -2 & 0 & 0 \\ 4 & 0 & 0 \\ -1 & 2 & 3 \end{bmatrix} \qquad R = \begin{bmatrix} 6 & 2 \\ 3 & -2 \\ -1 & 0 \end{bmatrix}$$

Solution.

$$A^\top = \begin{bmatrix} -1 & 0 & 6 \\ 4 & 1 & 0 \end{bmatrix}, \quad B^\top = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}, \quad C^\top = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}, \quad D^\top = \begin{bmatrix} -1, & 4, & 2 \end{bmatrix}$$

$$E^\top = \begin{bmatrix} 0 & 0 & -6 \\ 0 & -6 & 0 \\ 6 & 0 & 0 \end{bmatrix} \qquad F^\top = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} \qquad G^\top = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

$$H^\top = \begin{bmatrix} 0 & 1 & -6 & -2 \\ -1 & 0 & 7 & -1 \\ 6 & -7 & 0 & 4 \\ 2 & 1 & -4 & 0 \end{bmatrix} \qquad J^\top = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \qquad K^\top = \begin{bmatrix} 1 & -2 & -3 & -4 \\ 2 & 1 & -5 & -6 \\ 3 & 5 & 1 & -7 \\ 4 & 6 & 7 & 1 \end{bmatrix}$$

$$L^\top = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \qquad M^\top = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad N^\top = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P^\top = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad Q^\top = \begin{bmatrix} -2 & 4 & -1 \\ 0 & 0 & 2 \\ 0 & 0 & 3 \end{bmatrix} \qquad R^\top = \begin{bmatrix} 6 & 3 & -1 \\ 2 & -2 & 0 \end{bmatrix}$$

- Square matrices: $B, C, E, F, G, H, J, K, L, M, N, P, Q$
- Diagonal matrices: B, G, N
- Upper triangular matrices: B, G, L, N
- Lower triangular matrices: B, G, M, N, Q
- Symmetric matrices: B, F, G, J, N, P
- Skew-symmetric matrices: H

For Questions 8-13, let

$$A = \begin{bmatrix} -2 & 3 \\ 6 & 5 \\ 1 & -4 \end{bmatrix} \qquad B = \begin{bmatrix} -5 & 3 & 6 \\ 3 & 8 & 0 \\ -2 & 0 & 4 \end{bmatrix} \qquad C = \begin{bmatrix} 11 & -2 \\ -4 & -2 \\ 3 & -1 \end{bmatrix}$$

$$D = \begin{bmatrix} -1 & 4 & 3 & 7 \\ 2 & 1 & 7 & 5 \\ 0 & 5 & 5 & -2 \end{bmatrix} \qquad E = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \qquad F = \begin{bmatrix} 9 & -3 \\ 5 & -4 \\ 2 & 0 \\ 8 & -3 \end{bmatrix}$$

$$G = \begin{bmatrix} 5 & 1 & 0 \\ 0 & -2 & -1 \\ 1 & 0 & 3 \end{bmatrix} \qquad H = \begin{bmatrix} 6 & 3 & 1 \\ 1 & -15 & -5 \\ -2 & -1 & 10 \end{bmatrix} \qquad J = \begin{bmatrix} 8 \\ -1 \\ 4 \end{bmatrix}$$

$$K = \begin{bmatrix} 2 & 1 & -5 \\ 0 & 2 & 7 \end{bmatrix} \qquad L = \begin{bmatrix} 10 & 9 \\ 8 & 7 \end{bmatrix} \qquad M = \begin{bmatrix} 7 & -1 \\ 11 & 3 \end{bmatrix}$$

$$N = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad P = \begin{bmatrix} 3 & -1 \\ 4 & 7 \end{bmatrix} \qquad Q = \begin{bmatrix} 1 & 4 & -1 & 6 \\ 8 & 7 & -3 & 3 \end{bmatrix}$$

$$R = \begin{bmatrix} -3, & 6, & -2 \end{bmatrix} \qquad S = \begin{bmatrix} 6, & -4, & 3, & 2 \end{bmatrix} \qquad T = \begin{bmatrix} 4, & -1, & 7 \end{bmatrix}$$

Question 8. Compute, if possible, the following matrices.

1. AB
2. BA
3. JM
4. DF
5. RJ
6. JR
7. RT
8. SF
9. KN
10. F^2
11. B^2
12. E^3
13. $(TJ)^3$
14. $D(FK)$
15. $(CL)G$

Solution.

1. AB is not defined
2. $BA = \begin{bmatrix} 34 & -24 \\ 42 & 49 \\ 8 & -22 \end{bmatrix}$
3. JM is not defined
4. $DF = \begin{bmatrix} 73 & -34 \\ 77 & -25 \\ 19 & -14 \end{bmatrix}$
5. $RJ = \begin{bmatrix} -38 \end{bmatrix}$
6. JR is not defined

7. RT is not defined

8. $SF = \begin{bmatrix} 56, & -8 \end{bmatrix}$

9. KN is not defined

10. F^2 is not defined

11. $B^2 = \begin{bmatrix} 22 & 9 & -6 \\ 9 & 73 & 18 \\ 2 & -6 & 4 \end{bmatrix}$

12. $E^3 = \begin{bmatrix} 5 & 3 & 2 & 5 \\ 4 & 1 & 3 & 1 \\ 1 & 1 & 0 & 2 \\ 4 & 1 & 3 & 1 \end{bmatrix}$

13. $(TJ)^3 = \begin{bmatrix} 3721 \end{bmatrix}$

14. $D(FK) = \begin{bmatrix} 146 & 5 & -603 \\ 154 & 27 & -560 \\ 38 & -9 & -193 \end{bmatrix}$

15. $(CL)G$ is not defined

Question 9. Identify which of the following matrix pairs commute.

1. L and M
2. F and Q
3. G and H
4. R and J
5. A and K
6. N and P

Solution.

1. L and M do not commute

$$LM = \begin{bmatrix} 169 & 17 \\ 133 & 13 \end{bmatrix}, \quad ML = \begin{bmatrix} 62 & 56 \\ 134 & 120 \end{bmatrix}$$

2. F and Q are not square matrices, so they do not commute

3. H and G do not commute

$$GH = \begin{bmatrix} 31 & 0 & 0 \\ 0 & 31 & 0 \\ 0 & 0 & 31 \end{bmatrix}, \quad HG = \begin{bmatrix} 37 & -28 & 1 \\ 1 & 233 & 26 \\ -33 & -1 & 103 \end{bmatrix}$$

4. R and J are not square matrices, so they do not commute

5. A and K are not square matrices, so they do not commute

6. N and P commute because

$$NP = PN = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Question 10. Find the specified row or column of the matrix product for the given matrices.

1. The second row of the product BG
2. The third column of the product DE
3. The first column of the product SE

4. The third row of the product FQ

Solution.

1. The second row of BG is given by

$$\begin{bmatrix}3, & 8, & 0\end{bmatrix} G = \begin{bmatrix}15, & -13, & -8\end{bmatrix}$$

2. The third column of DE is given by

$$D\begin{bmatrix}0\\1\\0\\1\end{bmatrix} = \begin{bmatrix}11\\6\\3\end{bmatrix}$$

3. The first column of SE is given by

$$S\begin{bmatrix}1\\1\\0\\1\end{bmatrix} = \begin{bmatrix}11\\6\\3\end{bmatrix} = \begin{bmatrix}4\end{bmatrix}$$

4. The third row of FQ is given by

$$\begin{bmatrix}2, & 0\end{bmatrix} Q = \begin{bmatrix}2, & 8, & -2, & 12\end{bmatrix}$$

Question 11. Verify whether the following computations can be performed. If they are possible, identify which of the specified equalities hold. Provide justification by naming the relation and referencing the mathematical properties that confirm or disprove the validity of the equality.

1. $(RG)H = R(GH)$
2. $LP = PL$
3. $E(FK) = (EF)K$
4. $K(A + C) = KA + KC$
5. $(QF)^\top = F^\top Q^\top$
6. $L(ML) = L^2M$
7. $GC + HC = (G + H)C$
8. $R(J + T^\top) = RJ + RT^\top$
9. $(AK)^\top = A^\top K^\top$
10. $(Q + F^\top)E^\top = QE^\top + (EF)^\top$

Solution.

1. $(RG)H$ and $R(GH)$ are both well-defined. Furthermore, it follows from the associative law of matrix multiplication that $(RG)H = R(GH)$ We have

$$(RG)H = R(GH) = \begin{bmatrix} -93 & 186 & -62 \end{bmatrix}$$

2. LP and PL are both well-defined. But they are not equal

$$LP = \begin{bmatrix} 66 & 53 \\ 52 & 41 \end{bmatrix}, \quad PL = \begin{bmatrix} 22 & 20 \\ 96 & 85 \end{bmatrix}$$

3. $E(FK)$ and $(EF)K$ are both well-defined. Furthermore, it follows from the associative law of matrix multiplication that $E(FK) = (EF)K$ We have

$$E(FK) = (EF)K = \begin{bmatrix} 44 & 2 & -180 \\ 22 & 5 & -76 \\ 16 & 2 & -61 \\ 22 & 5 & -76 \end{bmatrix}$$

4. $K(A+C)$ and $KA+KC$ are both well-defined. Furthermore, it follows from the distributive law of matrix multiplication over addition that $K(A+C) = KA+KC$. We have

$$K(A+C) = KA+KC = \begin{bmatrix} 0 & 30 \\ 32 & -29 \end{bmatrix}$$

5. $(QF)^\top$ and $F^\top Q^\top$ are both well-defined. Furthermore, it follows from the definition of matrix transpose and matrix multiplication that $(QF)^\top = F^\top Q^\top$. We have

$$(QF)^\top = F^\top Q^\top = \begin{bmatrix} 75 & 125 \\ -37 & -61 \end{bmatrix}$$

6. $L(ML)$ and $L^2M$are both well-defined. However, they are not equal.

$$L(ML) = \begin{bmatrix} 1826 & 1640 \\ 1434 & 1288 \end{bmatrix}, \quad L^2M = \begin{bmatrix} 2887 & 287 \\ 2283 & 227 \end{bmatrix}.$$

7. $GC+HC$ and $(G+H)C$ are both well-defined. Furthermore, it follows from the distributive law of matrix multiplication over addition that $GC+HC = (G+H)C$. We have

$$GC+HC = (G+H)C = \begin{bmatrix} 108 & -31 \\ 61 & 38 \\ 32 & -9 \end{bmatrix}$$

8. $R(J+T^\top)$ and $RJ+RT^\top$ are both well-defined. Furthermore, it follows from the distributive law of matrix multiplication over addition that $R(J+T^\top) = RJ+RT^\top$. We have

$$R(J+T^\top) = RJ+RT^\top = \begin{bmatrix} -70 \end{bmatrix}.$$

9. $(AK)^\top$ and $A^\top K^\top$ are both well-defined. However, the equality does not hold. We have

$$(AK)^\top = \begin{bmatrix} -4 & 12 & 2 \\ 4 & 16 & -7 \\ 31 & 5 & -33 \end{bmatrix}, \quad A^\top K^\top = \begin{bmatrix} -3 & 19 \\ 31 & -18 \end{bmatrix}.$$

10. Both sides of the equality are well-defined. First, it follows from the distributive law of matrix multiplication over addition that $(Q+F^\top)E^\top = QE^\top + F^\top E^\top$. Then it follows from the definition of matrix transpose and matrix multiplication that $F^\top E^\top = (EF)^\top$. We have
$$(Q+F^\top)E^\top = QE^\top + (EF)^\top = \begin{bmatrix} 33 & 11 & 14 & 11 \\ 8 & 2 & 0 & 2 \end{bmatrix}$$

Question 12. Given matrices
$$A = \begin{bmatrix} 3 & -2 & 7 \\ 6 & 5 & 4 \\ 0 & 4 & 9 \end{bmatrix}, \quad B = \begin{bmatrix} 6 & -2 & 4 \\ 0 & 1 & 3 \\ 7 & 7 & 5 \end{bmatrix};$$
or
$$A = \begin{bmatrix} 1 & 5 & -2 & 1 \\ 0 & 1 & -2 & 3 \\ 1 & 1 & 2 & 1 \\ 0 & 0 & 2 & -4 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 & 5 & -1 \\ 2 & 1 & 2 & 1 \\ 4 & -2 & -1 & 0 \\ -1 & 2 & 1 & 3 \end{bmatrix}$$
Compute the specified row or column of the matrix product of the given matrices, both for the 3×3 and 4×4 cases:

1. The first row of the product AB
2. The third row of the product AB
3. The second column of the product AB
4. The first column of the product BA
5. The second column of the product BA
6. The third row of the product AA
7. The third column of the product AA
8. The first column of the product AA
9. The second column of the product BB
10. The second row of the product BB
11. The third column of the product BAA

Solution.

1. The first row of the product AB is given by
$$\begin{bmatrix} 3, & -2, & 7 \end{bmatrix} B = 3\begin{bmatrix} 6, & -2, & 4 \end{bmatrix} - 2\begin{bmatrix} 0, & 1, & 3 \end{bmatrix} + 7\begin{bmatrix} 7, & 7, & 5 \end{bmatrix} = \begin{bmatrix} 67 & 41 & 41 \end{bmatrix}$$
or
$$\begin{bmatrix} 1 & 5 & -2 & 1 \end{bmatrix} B = \begin{bmatrix} 1 & 12 & 18 & 7 \end{bmatrix}$$
2. The third row of the product AB is given by
$$\begin{bmatrix} 0, & 4, & 9 \end{bmatrix} B = \begin{bmatrix} 63 & 67 & 57 \end{bmatrix}$$
or
$$\begin{bmatrix} 1 & 1 & 2 & 1 \end{bmatrix} B = \begin{bmatrix} 9 & 0 & 6 & 3 \end{bmatrix}$$

3. The second column of the product AB is given by

$$A\begin{bmatrix}-2\\1\\7\end{bmatrix}=\begin{bmatrix}41\\21\\67\end{bmatrix}$$

or

$$A\begin{bmatrix}1\\1\\-2\\2\end{bmatrix}=\begin{bmatrix}12\\11\\0\\-12\end{bmatrix}$$

4. The first column of the product BA is given by

$$B\begin{bmatrix}3\\6\\0\end{bmatrix}=\begin{bmatrix}6\\6\\63\end{bmatrix}$$

or

$$B\begin{bmatrix}1\\0\\1\\0\end{bmatrix}=\begin{bmatrix}5\\4\\3\\0\end{bmatrix}$$

5. The second column of the product BA is given by

$$\begin{bmatrix}-6\\17\\41\end{bmatrix},\quad\text{or}\quad\begin{bmatrix}6\\13\\17\\-2\end{bmatrix}$$

6. The third row of the product AA is given by

$$\begin{bmatrix}24 & 56 & 97\end{bmatrix},\quad\text{or}\quad\begin{bmatrix}3 & 8 & 2 & 2\end{bmatrix}$$

7. The third column of the product AA is given by

$$\begin{bmatrix}76\\98\\97\end{bmatrix},\quad\text{or}\quad\begin{bmatrix}-14\\0\\2\\-4\end{bmatrix}$$

8. The first column of the product AA is given by

$$\begin{bmatrix}-3\\48\\24\end{bmatrix},\quad\text{or}\quad\begin{bmatrix}-1\\-2\\3\\2\end{bmatrix}$$

9. The second column of the product BB is given by

$$\begin{bmatrix}14\\22\\28\end{bmatrix},\quad\text{or}\quad\begin{bmatrix}-11\\1\\4\\5\end{bmatrix}$$

10. The second row of the product BB is given by

$$\begin{bmatrix} 21 & 22 & 18 \end{bmatrix}, \quad \text{or} \quad \begin{bmatrix} 9 & 1 & 11 & 2 \end{bmatrix}$$

11. The third column of the product BAA is given by

$$\begin{bmatrix} 648 \\ 389 \\ 1703 \end{bmatrix}, \quad \text{or} \quad \begin{bmatrix} 14 \\ -28 \\ -58 \\ 4 \end{bmatrix}$$

Question 13.

1. Find a non-diagonal matrix A such that $A^2 = I_2$.
2. Find a non-diagonal matrix A such that $A^2 = I_3$ (modify the result from part (a)).
3. Find a non-identity matrix A such that $A^3 = I_3$.

Solution.

1. Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, then we have

$$A^2 = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} a_{11}^2 + a_{12}a_{21} & a_{11}a_{12} + a_{12}a_{22} \\ a_{21}a_{11} + a_{22}a_{21} & a_{21}a_{12} + a_{22}^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Consequently,

$$a_{11}^2 + a_{12}a_{21} = 1, \ a_{11}a_{12} + a_{12}a_{22} = 0, \ a_{21}a_{11} + a_{22}a_{21} = 0, \ a_{21}a_{12} + a_{22}^2 = 1.$$

Since A is not diagonal, $a_{12} \neq 0$ or $a_{21} \neq 0$. Then

$$a_{11} = -a_{22}, \quad a_{11}^2 = 1 - a_{12}a_{21}.$$

If we let

$$a_{12} = a_{21} = 1, \quad a_{11} = a_{22} = 0,$$

the matrix

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

is one matrix that satisfies $A^2 = I_2$. Similarly,

$$a_{12} = a_{21} = \frac{1}{2}, \quad a_{11} = \frac{\sqrt{3}}{2}, \quad a_{22} = \frac{\sqrt{3}}{2}$$

gives

$$A = \frac{1}{2} \begin{bmatrix} \sqrt{3} & 1 \\ 1 & -\sqrt{3} \end{bmatrix}$$

2.

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

3.

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$