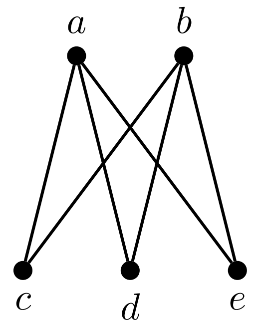
Tutorial 12

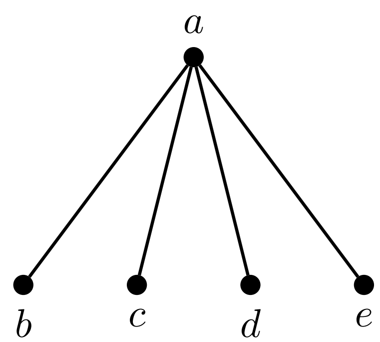
Matching

Question 1. Draw the complete bipartite graphs $K_{2,3}$, $K_{1,4}$, and $K_{3,5}$. *Solution.*

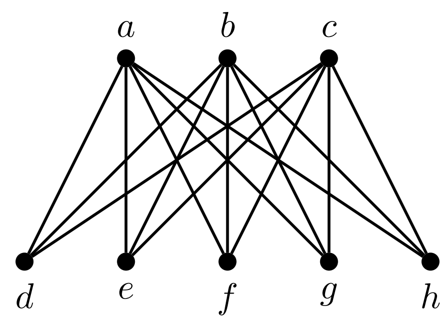
1. $K_{2,3}$



2. $K_{1,4}$



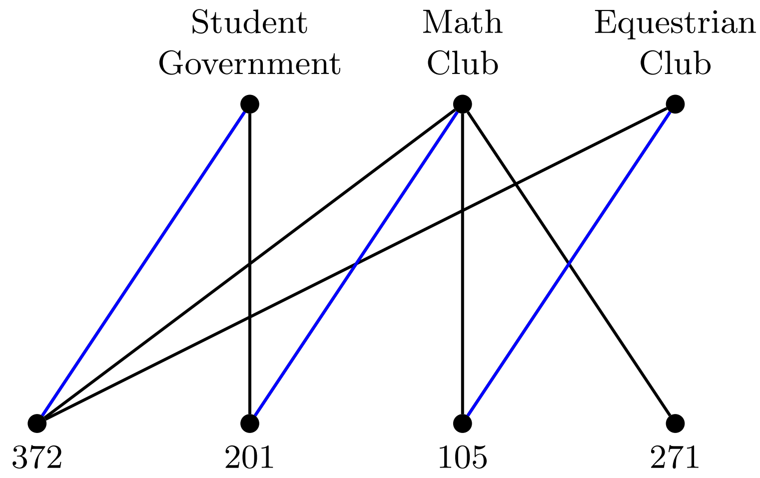
3. $K_{3,5}$



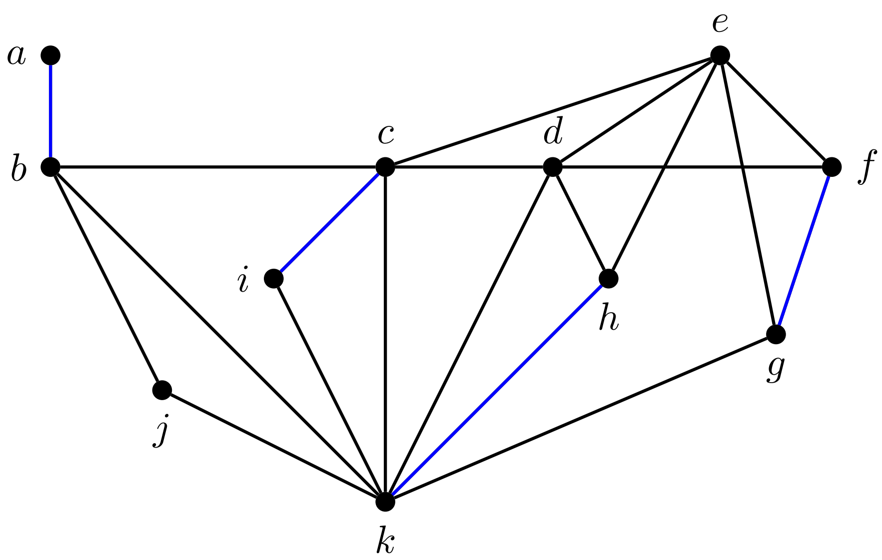
Question 2. Three student organizations (Student Government, Math Club, and the Equestrian Club) are holding meetings on Thursday afternoon. The only available rooms are 105, 201, 271, and 372. Based on membership and room size, the Student Government can only use

201 or 372, Equestrian Club can use 105 or 372, and Math Club can use any of the four rooms. Find a maximum matching for this scenario.

Solution.



Question 3. Below is a graph with a matching M shown in blue.

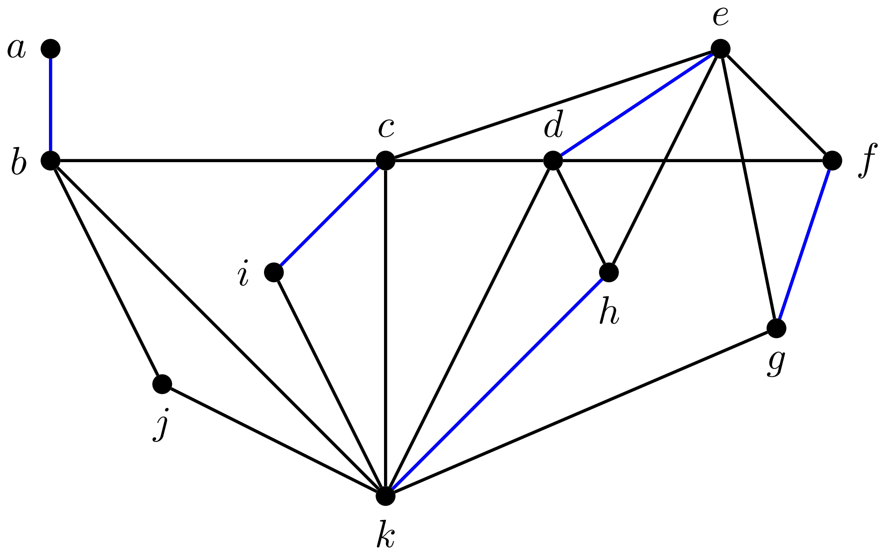


1. Find an alternating path starting at a. Is this path augmenting?
2. Find an augmenting path in the graph or explain why none exists.
3. Is M a maximum matching? maximal matching? perfect matching? Explain your answer. If M is not maximum, find a matching that is maximum.

Solution.

1. *abcikhe*, not augmenting
2. *jkhe*
3.
   - M is not maximum, since an augmenting path exists (Berge's Theorem)
   - It is not maximal, because we can add an edge ed

- Not perfect because it does not saturate all vertices
- A maximum matching



Question 4. Each of the graphs below has a matching shown in blue. Complete the following steps for both:

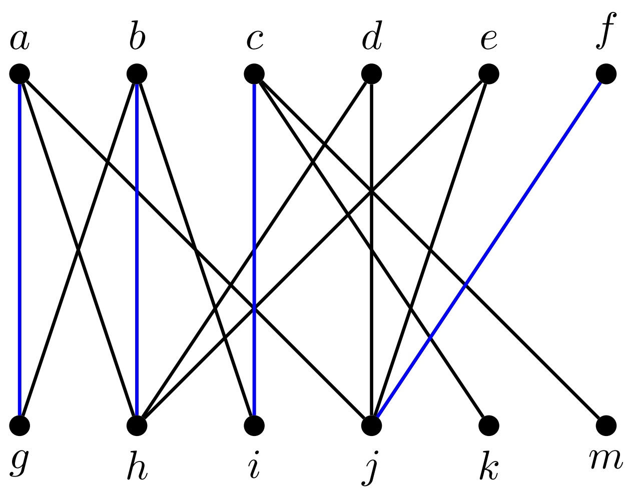
(i) Find an alternating path starting at vertex a.

(ii) Is this path augmenting? Explain your answer.

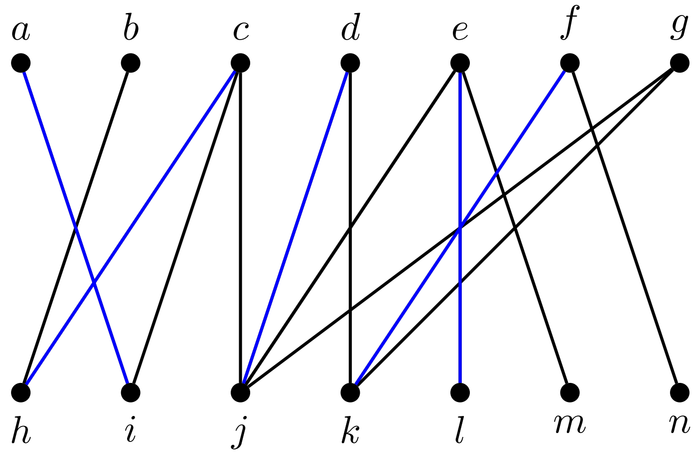
(iii) Use the Augmenting Path Algorithm to find a maximum matching.

(iv) Use the Vertex Cover Method to find a minimum vertex cover.

1.

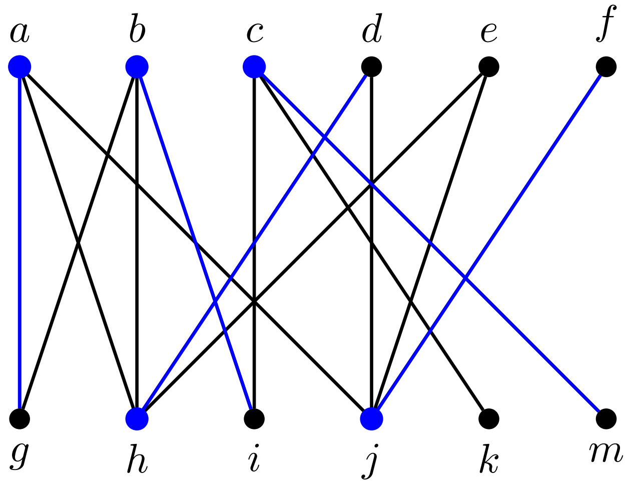


2.

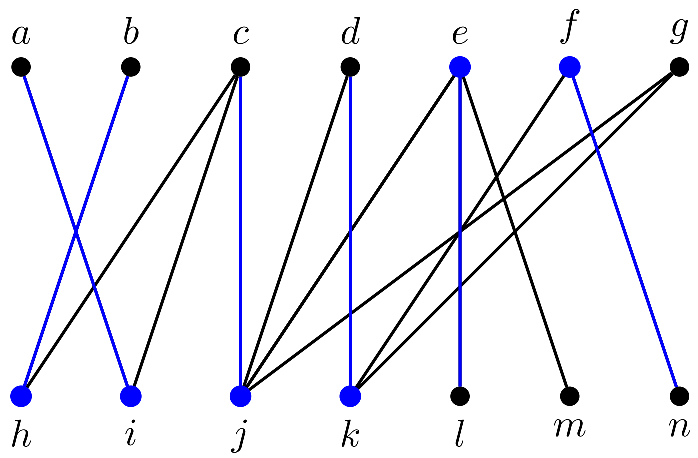


Solution.

1. *agbhd*; Not augmenting; Maximum matching and vertex cover

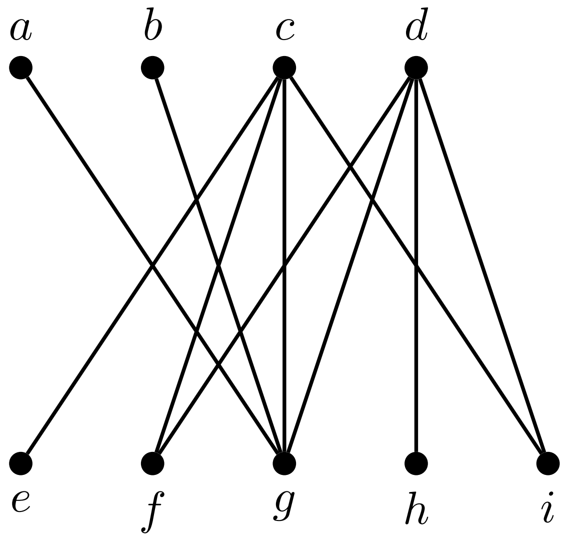


2. *agbicm*; Not augmenting; Maximum matching and vertex cover

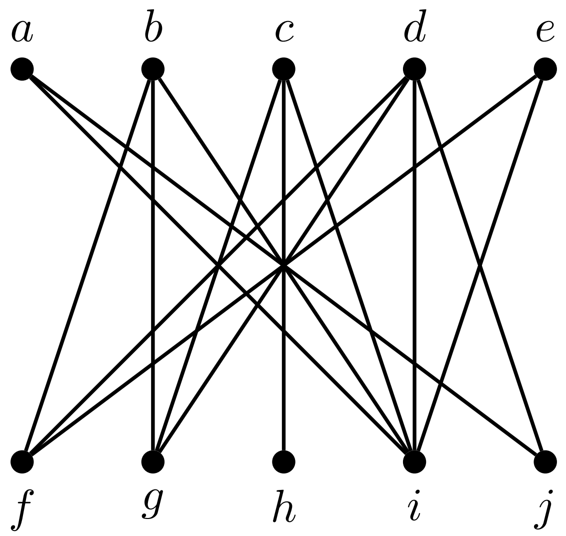


Question 5. Find a maximum matching for each of the graphs below. Include an explanation as to why the matching is maximum.

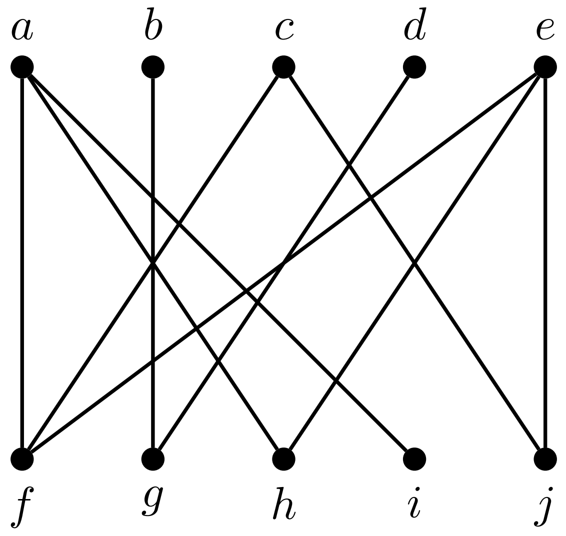
1.



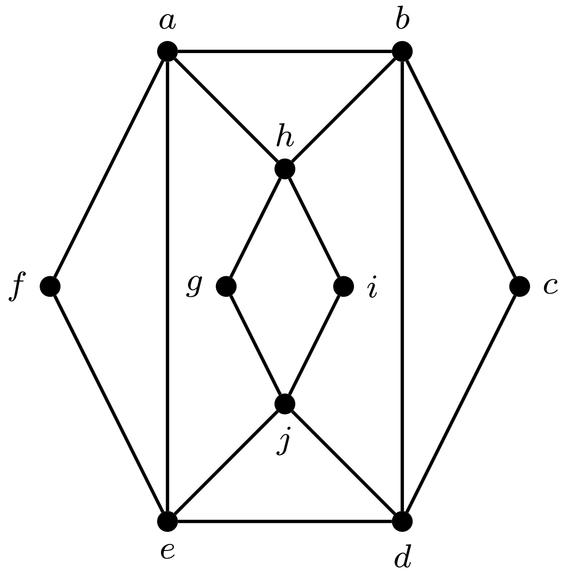
2.



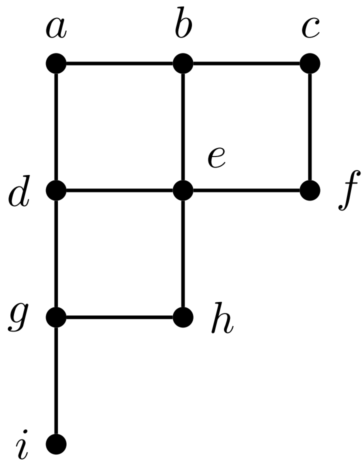
3.



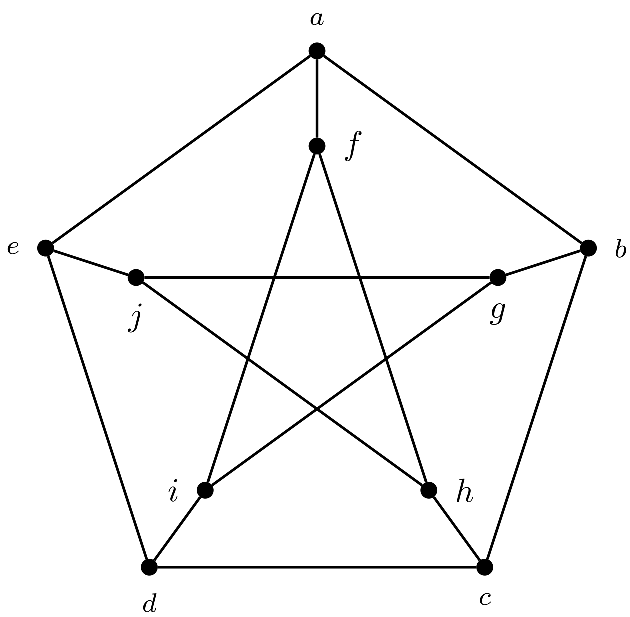
4.



5.



6.



Solution.

1. bg, ce, dh
2. bf, dg, aj, ch, ei, there are five vertices in each partition. Maximum possible size of a match is five.
3. bg, af, cj, eh, both b and d are only adjacent to g, maximum possible size of a match is four.
4. ab, fe, cd, hi, gj. There are in total 10 vertices, maximum possible size of a match is five.
5. ab, cf, ed, gh. There are in total 9 vertices, maximum possible size of a match is four.
6. af, bg, hc, di, ej. There are in total 10 vertices, maximum possible size of a match is five.

Question 6. Using the graphs from Question 5,

(i) Determine which graphs are bipartite.

(ii) For each of the graphs that are bipartite, find a minimum vertex cover. Verify that the size of the matching found in Question 5 equals the size of your vertex cover.

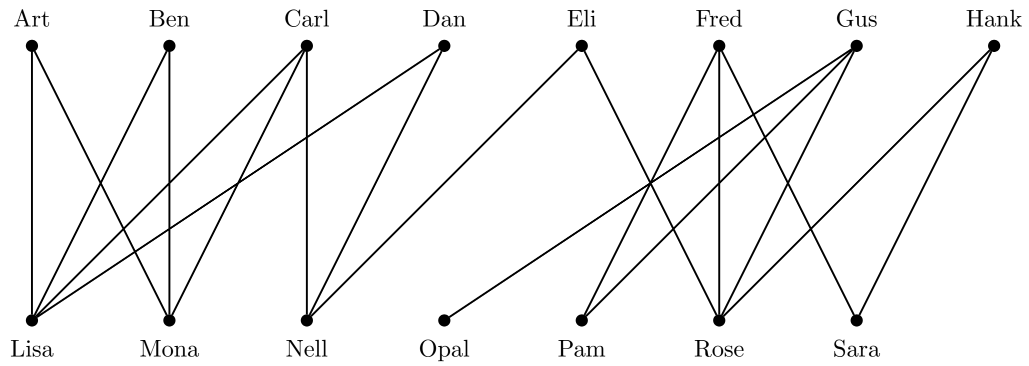
Solution.

(i) Bipartite graphs: $1, 2, 3, 5$

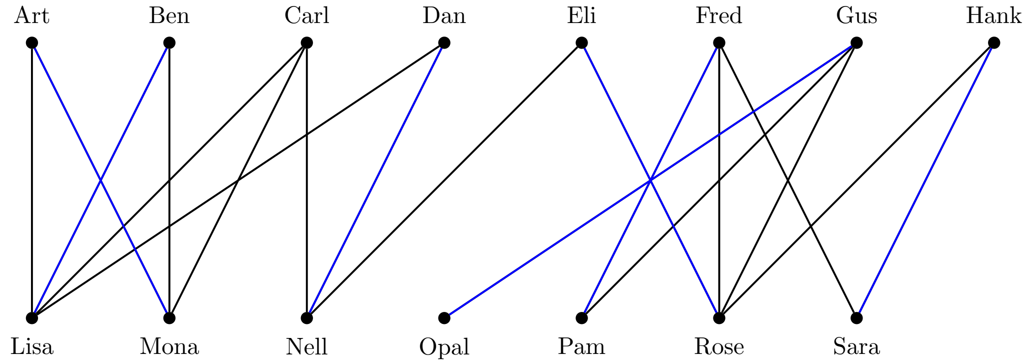
(ii) Vertex covers:

1. c, d, g
2. a, b, c, d, e
3. a, c, e, g
5. a, c, e, g

Question 7. The Roanoke Ultimate Frisbee League is organizing a Contra Dance. The fifteen members must be split into male-female pairs, though not all people are willing to dance with each other. The graph below models those who can be paired (as both people find the other acceptable). Find a maximum matching and explain why a larger matching does not exist.



Solution.



All vertices in the lower part are saturated.

Question 8.

Definition 1 *Given a collection of finite nonempty sets $S_1, S_2, \ldots, S_n$ (where $n \geq 1$), a system of distinct representatives is a collection $r_1, r_2, \ldots, r_n$ such that*

$$r_i \in S_i, \quad r_i \neq r_j \text{ if } i \neq j$$

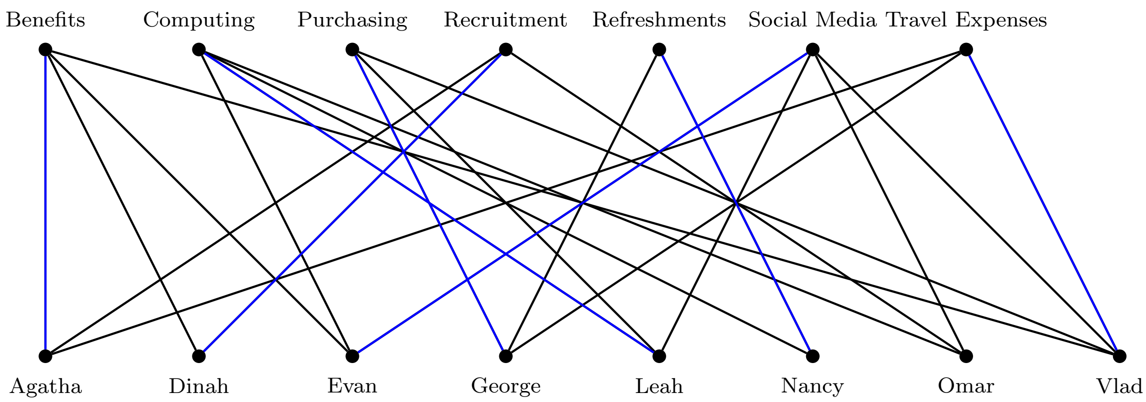
for all $i, j = 1, 2, \ldots, n$.

In less technical terms, the idea of distinct representatives is that a collection of groups each need their own representative and no two groups can have the same representative.

Seven committees must elect a chairperson to represent them at the end-of-year board meeting; however, some people serve on more than one committee and so cannot be elected chairperson for more than one committee. Based on the membership lists below, use a bipartite graph to determine a system of distinct representatives for the board meeting.

Committee	**Members**			
Benefits	Agatha	Dinah	Evan	Vlad
Computing	Evan	Nancy	Leah	Omar
Purchasing	George	Vlad	Leah	
Recruitment	Dinah	Omar	Agatha	
Refreshments	Nancy	George		
Social Media	Evan	Leah	Vlad	Omar
Travel Expenses	Agatha	Vlad	George	

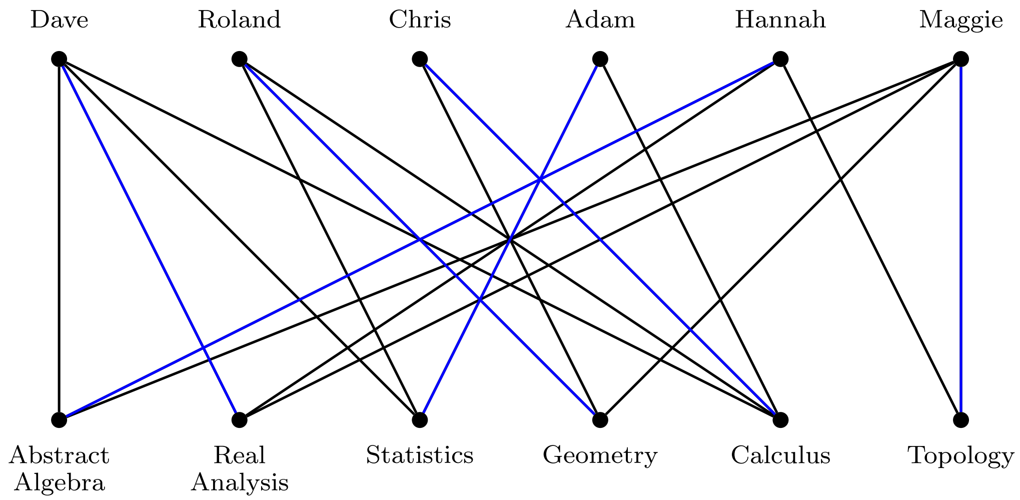
Solution. One possible solution: Benefits–Agatha, Computing–Leah, Purchasing– George, Recruitment–Dinah, Refreshments–Nancy, Social Media–Evan, Travel Expenses–Vlad.



Question 9. Each year, the chair of the mathematics department must determine course assignments for the faculty. Each professor has submitted a list of the courses he or she wants to teach. Find a system of assignments where each professor will teach exactly one of the remaining courses or explain why none exists.

Professor	**Preferred Courses**			
Dave	Abstract Algebra	Real Analysis	Statistics	Calculus
Roland	Statistics	Geometry	Calculus	
Chris	Calculus	Geometry		
Adam	Statistics	Calculus		
Hannah	Abstract Algebra	Real Analysis	Topology	
Maggie	Abstract Algebra	Real Analysis	Geometry	Topology

Solution. One possible solution: Dave-Real Analysis, Ronald-Geometry, Chris-Calculus, Adam-Statistics, Hannah-Abstract Algebra, Maggie-Topology



Question 10. Instead of pairing a professor with only one course of their preference from Question 9, now the mathematics department chair must pair each professor with two of the courses from their (expanded) list.

1. Describe how to turn this into a matching problem where a solution is given in terms of a perfect matching.

2. Find a perfect matching for the professors and their preferred course list shown below or explain why none exists.

Professor	**Preferred Courses**		
Dave	Abstract Algebra	Real Analysis	Number Theory
	Calculus II	Calculus I	Statistics
Roland	Vector Calculus	Discrete Math	Statistics
	Calculus II	Geometry	Calculus I
Chris	Vector Calculus	Real Analysis	Discrete Math
	Statistics	Geometry	Calculus I
Adam	Statistics	Calculus I	Number Theory
	Geometry	Differential Equations	
Hannah	Abstract Algebra	Real Analysis	Number Theory
	Linear Algebra	Topology	
Maggie	Abstract Algebra	Real Analysis	Linear Algebra
	Geometry	Topology	Calculus II

Solution. To model this as a perfect matching problem, we use the following approach:

- Vertices in X represent professor-copies for each professor, create two vertices
- Vertices in Y represent courses
- Draw and edge between a $x \in X$ and $y \in Y$ if y is on the preference list of x

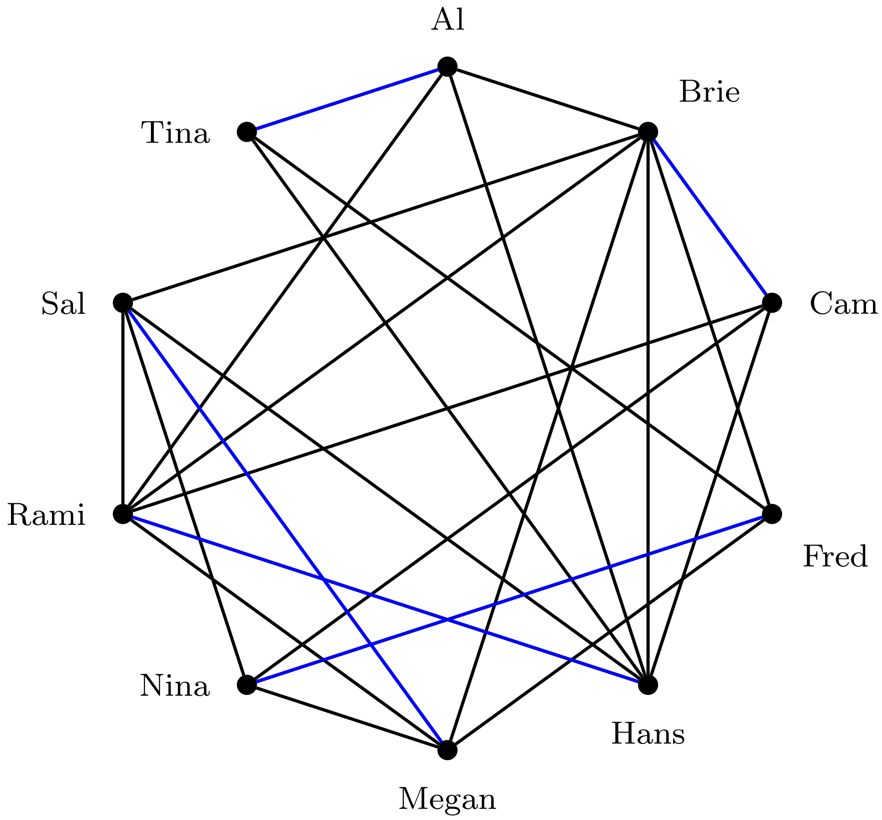
There are six professors, hence 12 vertices in X, and 12 courses, hence 12 vertices in Y. The solution to the problem is a perfect matching for the bipartite graph.
One possible perfect matching is:

- Dave - Abstract Algebra, Real Analysis
- Roland - Statistics, Discrete Math
- Chris - Calculus I, Vector Calculus
- Adam - Number Theory, Differential Equations
- Hannah - Linear Algebra, Topology
- Maggie - Geometry, Calculus II

Question 11. The students in a geometry course are paired each week to present homework solutions to the class. In the table below, a possible pair is indicated by a Y. Find a way to pair the students or explain why none exists.

	Al	Brie	Cam	Fred	Hans	Megan	Nina	Rami	Sal	Tina
Al	.	Y	.	.	Y	.	.	Y	.	Y
Brie	Y	.	Y	Y	Y	Y	.	Y	Y	.
Cam	.	Y	.	.	Y	.	Y	Y	.	.
Fred	.	Y	.	.	.	Y	Y	.	.	Y
Hans	Y	Y	Y	.	.	.	.	Y	Y	Y
Megan	.	Y	.	Y	.	.	Y	Y	Y	.
Nina	.	.	Y	Y	.	Y	.	.	Y	.
Rami	Y	Y	Y	.	Y	Y	.	.	Y	.
Sal	.	Y	.	.	Y	Y	Y	Y	.	.
Tina	Y	.	.	Y	Y	.	.	.	.	.

Solution.



Question 12. Apply the Gale–Shapley Algorithm to the set of preferences below with

1. the men proposing
2. the women proposing

Alice:	r > s > t > v	Rich:	a > d > b > c
Beth:	s > r > v > t	Stefan:	a > c > d > b
Cindy:	v > t > r > s	Tom:	c > b > d > a
Dahlia:	t > v > s > r	Victor:	c > d > b > a

Solution.

1. Rich-Alice, Victor-Cindy, Stefan-Dahlia, Tom-Beth
2. Alice-Rich, Beth-Stefan, Cindy-Victor, Dahlia-Tom

Question 13. Apply the Gale–Shapley Algorithm to the set of preferences below with

1. the men proposing
2. the women proposing

Edith:	l > n > o > m > p	Liam:	f > e > h > g > i
Faye:	n > l > m > o > p	Malik:	e > i > g > f > h
Grace:	p > m > o > n > l	Nate:	f > g > i > h > e
Hanna:	p > n > o > l > m	Olaf:	i > e > f > g > h
Iris:	p > o > m > n > l	Pablo:	f > h > g > e > i

Solution.

1. Nate-Faye, Liam-Edith, Pablo-Hanna, Olaf-Iris, Malik-Grace
2. Faye-Nate, Edith-Liam, Grace-Malik, Hanna-Pablo, Iris-Olaf

Question 14. Apply the Gale–Shapley Algorithm (with Unacceptable Partners) to the preferences from lecture with the women proposing

Anne:	t > r > w	Rob:	a > b > c > d
Brenda:	w > r > t	Stan:	a > b
Carol:	w > r > s > t	Ted:	c > d > a > b
Diana:	s > r > t	Will:	c > b > a

Solution. Carol-Will, Anne-Rob, Diana-Ted

Question 15. Apply the Gale–Shapley Algorithm (with Unacceptable Partners) to the preferences below with

1. the men proposing
2. the women proposing

Edith:	l > n > m	Liam:	f > e > h > g
Faye:	n > l > m > o > p	Malik:	e > h > i > f
Grace:	m > o > n > l	Nate:	g > f > i
Hanna:	p > o > l > m	Olaf:	i > e > f
Iris:	p > m > n > l	Pablo:	f > h > g > i

Solution.

1. Pablo-Hanna, Liam-Faye, Malik-Edith, Nate-Grace
2. Faye-Liam, Grace-Nate, Edith-Malik, Hanna-Pablo

Question 16. In each of the examples where the Gale-Shapley Algorithm is utilized, we have required that the number of men equals the number of women. Just as we were able to modify the algorithm for instances where some people are deemed unacceptable, we can modify the algorithm to account for unequal numbers. To do this, we introduce ghost participants in order to equalize the gender groups. These ghosts are deemed unacceptable by those of the opposite sex, and in turn find no person of the opposite sex acceptable. Using this modification, find a stable set of marriages for the preferences listed below.

Alice:	p > r > s > t	Peter:	b > a > c > d > e
Beth:	r > p > s > t	Rich:	c > b > e > d > a
Carol:	t > p > s > r	Saul:	a > b > c > d > e
Diana:	t > s > r > p	Teddy:	e > c > d > a > b
Edith:	r > s > t > p		

Solution.

1. Men proposing: Peter-Beth, Rich-Carol, Saul-Alice, Teddy-Edith
2. Women proposing: Edith-Teddy, Alice-Peter, Beth-Rich, Carol-Saul