Algebra and Discrete Mathematics (ADM)

Tutorial 2

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Matrices

- $\mathbb{R}$: the set of all real numbers

Definition

A *matrix with coefficients in* $\mathbb{R}$ is a rectangular array where each entry is an element of $\mathbb{R}$.

Matrix A is said to have m *rows*, n *columns* and is of size $m \times n$.

$$A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ a_{21} & \dots & a_{2n} \\ & \vdots & \\ a_{m1} & \dots & a_{mn} \end{bmatrix}.$$

Vectors

- A $1 \times n$ matrix is called a *row vector*.
- An $n \times 1$ matrix is called a *column vector*.

Note

- By "vector," we refer specifically to a row vector.
- $\mathbb{R}^n$ represents the set of all vectors with n entries, also referred to as *coordinates*.
- When written by hand, $\vec{a}$ is used to denote a vector.

Definition

The *norm* (also called *length*) of a vector $\boldsymbol{a} = \begin{bmatrix} a_1, & a_2, & \cdots, & a_n \end{bmatrix}$, denoted $\|\boldsymbol{a}\|$, is given by

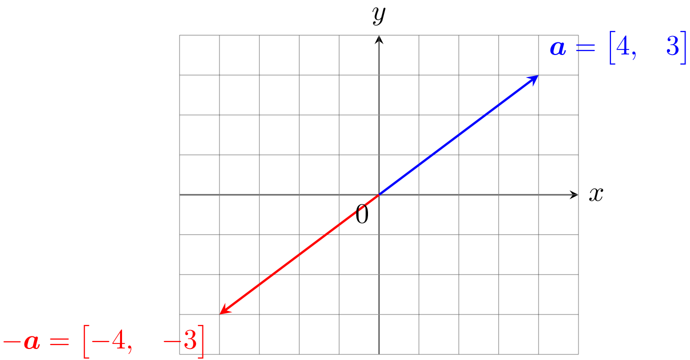
$$\|\boldsymbol{a}\| = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}.$$

A vector of norm 1 is called a *unit vector*

Vector – example

$$\boldsymbol{a} = \begin{bmatrix} 4, & 3 \end{bmatrix},$$
$$-\boldsymbol{a} = \begin{bmatrix} -4, & -3 \end{bmatrix}$$

$$\|\boldsymbol{a}\| = \sqrt{4^2 + 3^2} = \sqrt{16 + 9} = 5$$



Vector addition and subtraction

$$\boldsymbol{a} = \begin{bmatrix} 1, & -3, & 2, & 5 \end{bmatrix}, \boldsymbol{b} = \begin{bmatrix} 2, & 2, & 4, & 0 \end{bmatrix}$$

$$\begin{aligned} \boldsymbol{a} + \boldsymbol{b} &= ? \\ \boldsymbol{a} - \boldsymbol{b} &= ? \\ \boldsymbol{b} - \boldsymbol{a} &= ? \end{aligned}$$

Vector addition and subtraction

$$\boldsymbol{a} = \begin{bmatrix}1, & -3, & 2, & 5\end{bmatrix}, \boldsymbol{b} = \begin{bmatrix}2, & 2, & 4, & 0\end{bmatrix}$$

$$\begin{aligned}
\boldsymbol{a} + \boldsymbol{b} &= \boldsymbol{b} + \boldsymbol{a} = \begin{bmatrix}1+2, & -3+2, & 2+4, & 5+0\end{bmatrix} = \begin{bmatrix}3, & -1, & 6, & 5\end{bmatrix} \\
\boldsymbol{a} - \boldsymbol{b} &= \begin{bmatrix}1-2, & -3-2, & 2-4, & 5-0\end{bmatrix} = \begin{bmatrix}-1, & -5, & -2, & 5\end{bmatrix} \\
\boldsymbol{b} - \boldsymbol{a} &= \begin{bmatrix}1, & 5, & 2, & -5\end{bmatrix} = -(\boldsymbol{a} - \boldsymbol{b})
\end{aligned}$$

Projection vectors

- Projection of $\boldsymbol{a}$ onto $\boldsymbol{b}$ is given by

$$\text{proj}_{\boldsymbol{b}}\boldsymbol{a} = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{\|\boldsymbol{b}\|^2}\boldsymbol{b}$$

- Projection of $\boldsymbol{b}$ onto $\boldsymbol{a}$ is given by

$$\text{proj}_{\boldsymbol{a}}\boldsymbol{b} = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{\|\boldsymbol{a}\|^2}\boldsymbol{a}$$

Example

$\boldsymbol{a} = \begin{bmatrix} 1, & 3 \end{bmatrix}$, $\boldsymbol{b} = \begin{bmatrix} 5, & 1 \end{bmatrix}$

$$\text{proj}_{\boldsymbol{b}}\boldsymbol{a} = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{\|\boldsymbol{b}\|^2}\boldsymbol{b} \quad = \quad ?$$

$$\text{proj}_{\boldsymbol{a}}\boldsymbol{b} = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{\|\boldsymbol{a}\|^2}\boldsymbol{a} \quad = \quad ?$$

Projection vectors

- Projection of $\boldsymbol{a}$ onto $\boldsymbol{b}$ is given by

$$\mathrm{proj}_{\boldsymbol{b}}\boldsymbol{a} = \frac{\boldsymbol{a}\cdot\boldsymbol{b}}{\|\boldsymbol{b}\|^2}\boldsymbol{b}$$

- Projection of $\boldsymbol{b}$ onto $\boldsymbol{a}$ is given by

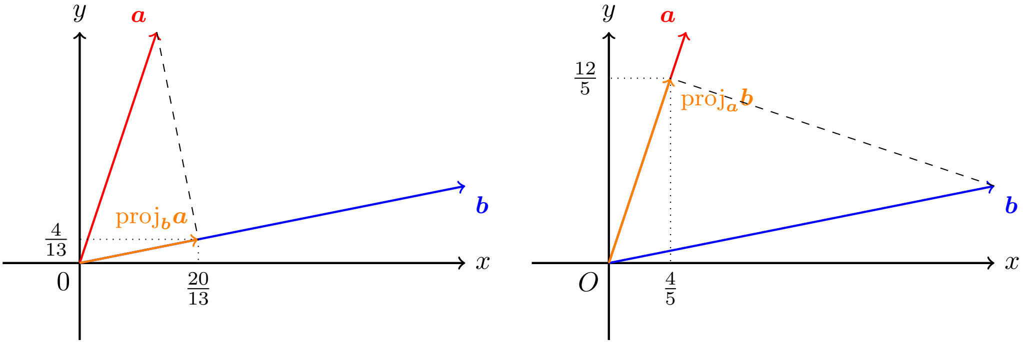
$$\mathrm{proj}_{\boldsymbol{a}}\boldsymbol{b} = \frac{\boldsymbol{a}\cdot\boldsymbol{b}}{\|\boldsymbol{a}\|^2}\boldsymbol{a}$$

Example

$\boldsymbol{a} = \begin{bmatrix}1, & 3\end{bmatrix}$, $\boldsymbol{b} = \begin{bmatrix}5, & 1\end{bmatrix}$

$$\begin{aligned}
\mathrm{proj}_{\boldsymbol{b}}\boldsymbol{a} = \frac{\boldsymbol{a}\cdot\boldsymbol{b}}{\|\boldsymbol{b}\|^2}\boldsymbol{b} \quad &= \quad \frac{1\times5+3\times1}{5^2+1}\boldsymbol{b} = \frac{8}{26}\begin{bmatrix}5, & 1\end{bmatrix} = \begin{bmatrix}\frac{20}{13}, & \frac{4}{13}\end{bmatrix} \\
\mathrm{proj}_{\boldsymbol{a}}\boldsymbol{b} = \frac{\boldsymbol{a}\cdot\boldsymbol{b}}{\|\boldsymbol{a}\|^2}\boldsymbol{a} \quad &= \quad \frac{1\times5+3\times1}{1^2+3^2}\boldsymbol{a} = \frac{8}{10}\begin{bmatrix}1, & 3\end{bmatrix} = \begin{bmatrix}\frac{4}{5}, & \frac{12}{5}\end{bmatrix}
\end{aligned}$$

Projection vectors



$$\boldsymbol{a} = \begin{bmatrix} 1, & 3 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 5, & 1 \end{bmatrix}, \quad \text{proj}_{\boldsymbol{b}}\boldsymbol{a} = \begin{bmatrix} \frac{20}{13}, & \frac{4}{13} \end{bmatrix}, \quad \text{proj}_{\boldsymbol{a}}\boldsymbol{b} = \begin{bmatrix} \frac{4}{5}, & \frac{12}{5} \end{bmatrix}$$

Matrices

$$A = [1] \in \mathcal{M}_{1\times 1}, \quad B = \begin{bmatrix}1, & 2\end{bmatrix} \in \mathcal{M}_{1\times 2}, \quad C = \begin{bmatrix}3\\5\end{bmatrix} \in \mathcal{M}_{2\times 1}, \quad D = \begin{bmatrix}1 & 2\\3 & 4\end{bmatrix} \in \mathcal{M}_{2\times 2}$$

$$E = \begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\end{bmatrix} \in \mathcal{M}_{2\times 3}$$

Special matrices

upper triangular matrix $A = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{bmatrix}$

lower triangular matrix $B = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 0 & -3 \end{bmatrix}$

diagonal matrix $C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad D = \begin{bmatrix} d_{11} & 0 & \dots & 0 \\ 0 & d_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_{nn} \end{bmatrix}$

zero matrix $O = \begin{bmatrix} 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}$

Transpose of a matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \quad A^\top = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

Matrix addition

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 0 & 2 \\ 3 & 5 & 1 \\ -2 & 2 & -3 \end{bmatrix}$$

$$A + B = ?$$

Matrix addition

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 0 & 2 \\ 3 & 5 & 1 \\ -2 & 2 & -3 \end{bmatrix}$$

$$A + B = B + A = \begin{bmatrix} 1 + (-1) & 2 + 0 & 3 + 2 \\ 4 + 3 & 5 + 5 & 6 + 1 \\ 7 + (-2) & 8 + 2 & 9 + (-3) \end{bmatrix} = \begin{bmatrix} 0 & 2 & 5 \\ 7 & 10 & 7 \\ 5 & 10 & 6 \end{bmatrix}$$

Matrix subtraction

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 0 & 2 \\ 3 & 5 & 1 \\ -2 & 2 & -3 \end{bmatrix}$$

$$A - B = ?$$

Matrix subtraction

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 0 & 2 \\ 3 & 5 & 1 \\ -2 & 2 & -3 \end{bmatrix}$$

$$A - B = -(B - A) = \begin{bmatrix} 1 - (-1) & 2 - 0 & 3 - 2 \\ 4 - 3 & 5 - 5 & 6 - 1 \\ 7 - (-2) & 8 - 2 & 9 - (-3) \end{bmatrix} = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 0 & 5 \\ 9 & 6 & 12 \end{bmatrix}$$

Matrix multiplication

$A \in \mathcal{M}_{m\times n}$, $B \in \mathcal{M}_{n\times r}$, $C = AB \in \mathcal{M}_{m\times r}$

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 3 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & -2 \\ -4 & 5 & 6 \end{bmatrix}$$

$$AB = ?$$

$$BA = ?$$

Matrix multiplication

$A \in \mathcal{M}_{m\times n}$, $B \in \mathcal{M}_{n\times r}$, $C = AB \in \mathcal{M}_{m\times r}$

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 3 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & -2 \\ -4 & 5 & 6 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1\times 1 + 1\times(-4) & 1\times 0 + 1\times 1.5 & 1\times(-2) + 1\times 6 \\ 2\times 1 + 2\times(-4) & 2\times 0 + 2\times 1.5 & 2\times(-2) + 2\times 6 \\ 3\times 1 + 3\times(-4) & 3\times 0 + 3\times 1.5 & 3\times(-2) + 3\times 6 \end{bmatrix} = \begin{bmatrix} -3 & 5 & 4 \\ -6 & 10 & 8 \\ -9 & 15 & 12 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1\times 1 + 0\times 2 + 2\times 3 & 1\times 1 + 0\times 2 + (-2)\times 3 \\ -4\times 1 + 5\times 2 + 6\times 3 & -4\times 1 + 5\times 2 + 6\times 3 \end{bmatrix} = \begin{bmatrix} -5 & -5 \\ 24 & 24 \end{bmatrix}$$

Matrix multiplication

$$C = \begin{bmatrix} 1 & -1 \\ 5 & 3 \end{bmatrix}$$

$$C^2 = ?$$

$$CI_2 = ?$$

$$I_2C = ?$$

Matrix multiplication

$$C = \begin{bmatrix} 1 & -1 \\ 5 & 3 \end{bmatrix}$$

$$C^2 = \begin{bmatrix} 1 & -1 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 5 & 3 \end{bmatrix} = \begin{bmatrix} 1 \times 1 + (-1) \times 5 & 1 \times (-1) + (-1) \times 3 \\ 5 \times 1 + 3 \times 5 & 5 \times (-1) + 3 \times 3 \end{bmatrix} = \begin{bmatrix} -4 & -4 \\ 20 & 4 \end{bmatrix}$$

$$CI_2 = \begin{bmatrix} 1 & -1 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 \times 1 + (-1) \times 0 & 1 \times 0 + (-1) \times 1 \\ 5 \times 1 + 3 \times 0 & 5 \times 0 + 3 \times 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 5 & 3 \end{bmatrix}$$

$$I_2C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 5 & 3 \end{bmatrix} = \begin{bmatrix} 1 \times 1 + 0 \times 5 & 1 \times (-1) + 0 \times 3 \\ 0 \times 1 + 1 \times 5 & 0 \times (-1) + 1 \times 3 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 5 & 3 \end{bmatrix}$$

Matrix multiplication

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 0 & 0 \\ -3 & 1 & 0 \\ 4 & 5 & 2 \end{bmatrix}, \quad AB \neq BA$$

$$\begin{aligned} AB &= \begin{bmatrix} 1\times 3 + 1\times(-3) + (-1)\times 4 & 1\times 0 + 1\times 1 + (-1)\times 5 & 1\times 0 + 1\times 0 + (-1)\times 2 \\ 0\times 3 + 1\times(-3) + 2\times 4 & 0\times 0 + 1\times 1 + 2\times 5 & 0\times 0 + 1\times 0 + 2\times 2 \\ 0\times 3 + 0\times(-3) + 1\times 4 & 0\times 0 + 0\times 1 + 1\times 5 & 0\times 0 + 0\times 0 + 1\times 2 \end{bmatrix} \\ &= \begin{bmatrix} -4 & -4 & -2 \\ 5 & 11 & 4 \\ 4 & 5 & 2 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} BA &= \begin{bmatrix} 3\times 1 + 0\times 0 + 0\times 0 & 3\times 1 + 0\times 1 + 0\times 0 & 3\times(-1) + 0\times 2 + 0\times 1 \\ -3\times 1 + 1\times 0 + 0\times 0 & -3\times 1 + 1\times 1 + 0\times 0 & -3\times(-1) + 1\times 2 + 0\times 1 \\ 4\times 1 + 5\times 0 + 2\times 0 & 4\times 1 + 5\times 1 + 2\times 0 & 4\times(-1) + 5\times 2 + 2\times 1 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 3 & -3 \\ -3 & -2 & 5 \\ 4 & 8 & 8 \end{bmatrix} \end{aligned}$$