Algebra and Discrete Mathematics (ADM)

Tutorial 8 Eulerian tours

Lecturer: Bc. Xiaolu Hou, PhD.
xiaolu.hou@stuba.sk

Graph

Definition

A *graph* consists of two sets: $V(G)$, called the *vertex set*, and $E(G)$, called the edge set. An *edge*, denoted xy, is an unordered pair of vertices $x, y \in E(G)$.

Theorem (Handshaking Lemma)

Let $G = (V, E)$ *be a graph. Suppose* $V = \{v_1, v_2, \ldots, v_n\}$, *then*

$$\deg(v_1) + \deg(v_2) + \cdots + \deg(v_n) = 2|E|$$

Draw a graph

Question

Draw a graph with 6 vertices such that their degrees are

$$1,\ 2,\ 3,\ 4,\ 5,\ 2$$

We have

$$\sum_v \deg(v) = 1+2+3+4+5+2 = 17$$

is odd. Thus, such a graph does not exist

Draw a graph

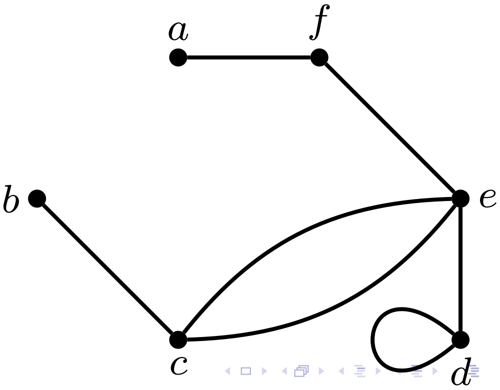
Question

Draw a graph with 6 vertices such that their degrees are

$$1, \ 2, \ 3, \ 4, \ 3, \ 1$$

$$\deg(a) = 1, \quad \deg(f) = 2, \quad \deg(c) = 3,$$

$$\deg(e) = 4, \quad \deg(d) = 3, \quad \deg(b) = 1,$$



Draw a graph

Question

Construct a graph to represent a club with 13 members, where each member is acquainted with exactly 5 other members.

The sum of degrees is given by

$$13 \times 5 = 65$$

is odd, such a graph does not exist.

Non-existence of certain regular graphs

Statement

Prove that a k-regular graph, where k is odd, with an odd number of vertices does not exist.

Proof.

The sum of the degrees of all vertices in a graph is given by:

$$\sum \text{degree} = (\text{number of vertices}) \times k.$$

Since both the number of vertices and k are odd, their product is also odd.
By the handshaking lemma, such a graph cannot exist. □

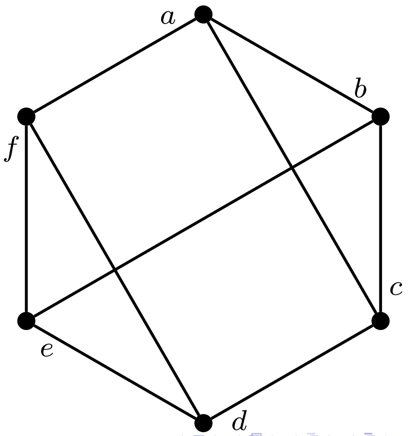
Draw a graph

Question

Draw a graph with 6 vertices, each of degree 3

$$\frac{6 \times 3}{2} = 9,$$

we know that the graph has 9 edges



Draw a graph

Question

Draw a graph with 7 vertices, each of degree 3

$$7 \times 3 = 21$$

is odd, such a graph does not exist.

Draw a graph

Question

Draw a graph with 4 vertices, each of degree 1

Note: the graph is not connected

Draw a graph

Question

Draw a graph with 6 edges, 4 vertices with degrees

$$1, \ 2, \ 3, \ 4$$

It must hold that

$$2 \times \text{number of edges} = \sum \text{degree}$$

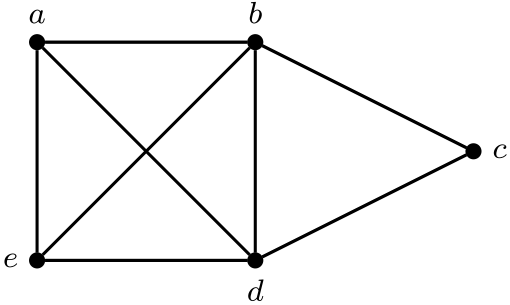
However

$$2 \times \text{number of edges} = 12$$

$$\sum \text{degree} = 1 + 2 + 3 + 4 = 10$$

The graph does not exist

Touring a graph



- Path: does not repeat vertices
  - Example: $a,\ b,\ c,\ d,\ e$
- Trail: does not repeat edges
  - Example: $a,\ b,\ e,\ a,\ d,\ c$
- Cycle: a closed path
  - Example: $a,\ b,\ c,\ d,\ e,\ a$
- Circuit: a closed trail
  - Example: $a,\ b,\ c,\ d,\ b,\ e,\ a$

Eulerian and semi-Eulerian

Theorem

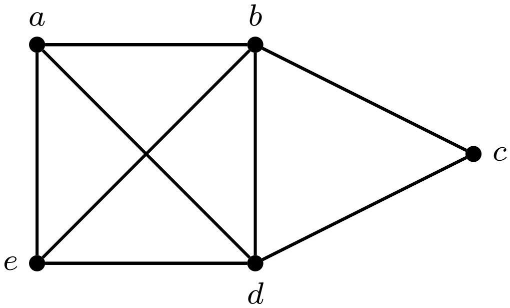
A graph G is Eulerian if and only if

- *G is connected and*
- *every vertex has an even degree*

A graph G is semi-Eulerian if and only if

- *G is connected and*
- *exactly two vertices have odd degree*

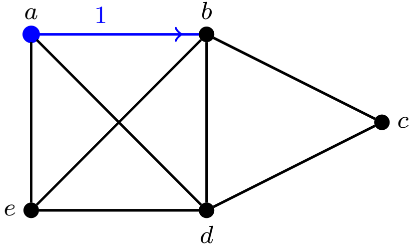
Eulerian circuit

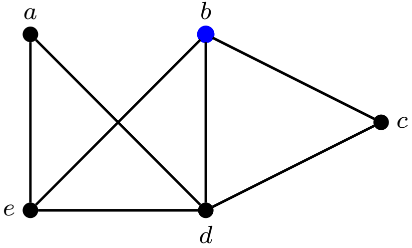


$$\deg(a) = 3, \quad \deg(b) = 4, \quad \deg(c) = 2, \quad \deg(d) = 4, \quad \deg(e) = 3$$

The graph is semi-Eulerian

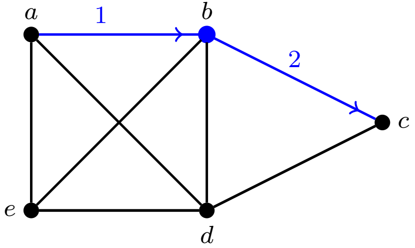
Fleury's Algorithm

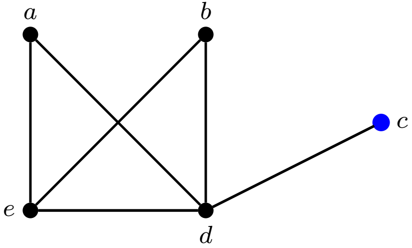




- Step 1. start vertex a
- Step 2. choose edge ab
- Step 3. travel to vertex b and delete edge ab. Current vertex: b.

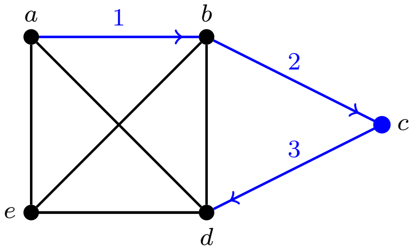
Fleury's Algorithm

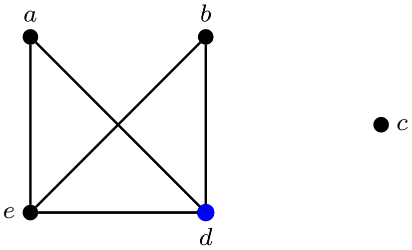




- Step 2. choose edge bc
- Step 3. travel to vertex c and delete edge bc. Current vertex: c.

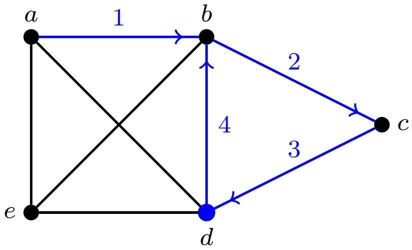
Fleury's Algorithm

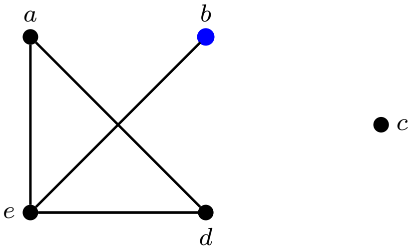




- Step 2. only choice edge cd
- Step 3. travel to vertex d and delete edge cd. Current vertex: d.

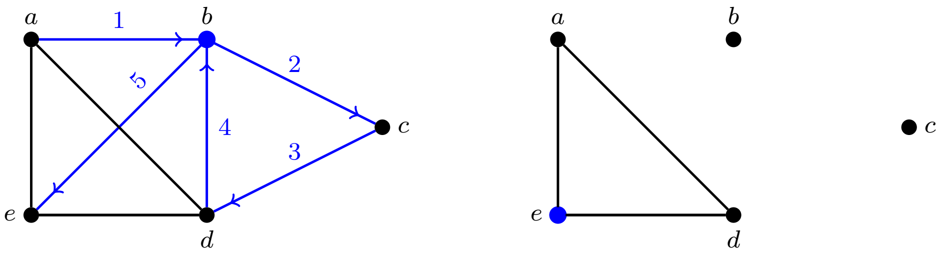
Fleury's Algorithm





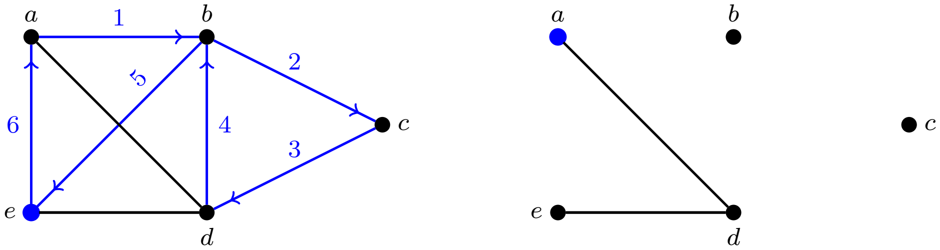
- Step 2. choose edge db
- Step 3. travel to vertex b and delete edge db. Current vertex: b.

Fleury's Algorithm



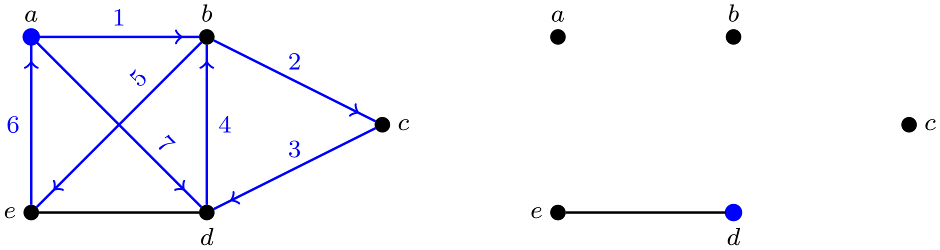
- Step 2. only choice be
- Step 3. travel to vertex e and delete edge be. Current vertex: e.

Fleury's Algorithm



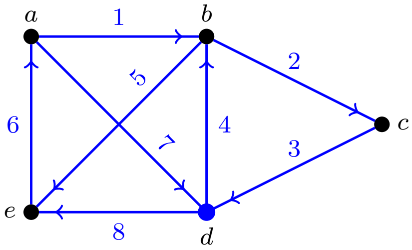
- Step 2. choose ea
- Step 3. travel to vertex a and delete edge ea. Current vertex: a.

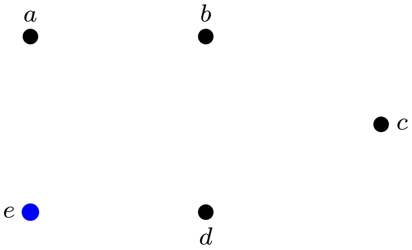
Fleury's Algorithm



- Step 2. only choice ad
- Step 3. travel to vertex d and delete edge ad. Current vertex: d.

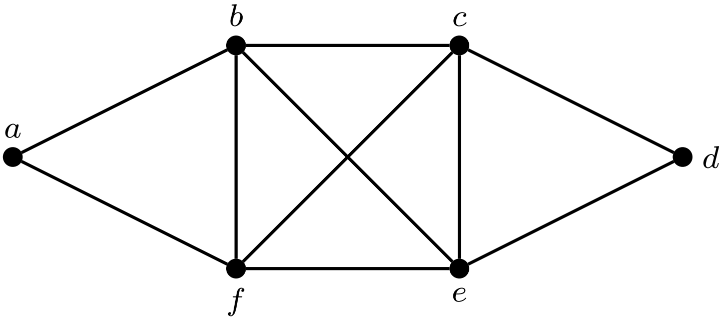
Fleury's Algorithm





- Step 2. only choice de
- Step 3. travel to vertex e and delete edge de.
- We have obtained an Eulerian circuit

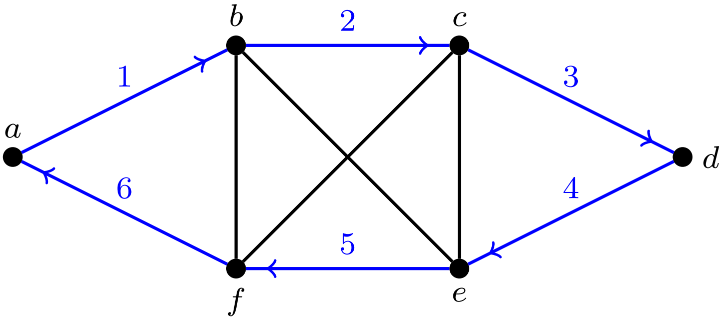
Hierholzer's Algorithm



$$\deg(a) = 2, \quad \deg(b) = 4, \quad \deg(c) = 4, \quad \deg(d) = 2, \quad \deg(e) = 4, \quad \deg(f) = 4$$

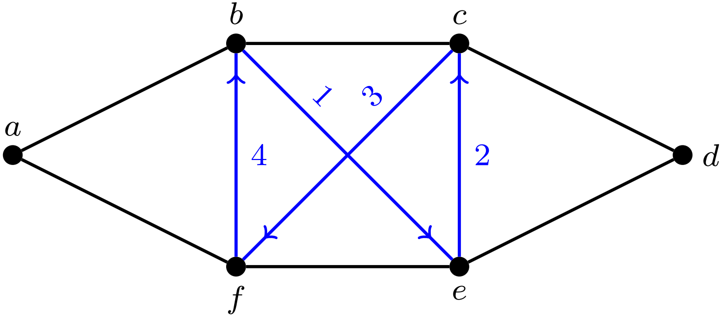
The graph is Eulerian

Hierholzer's Algorithm



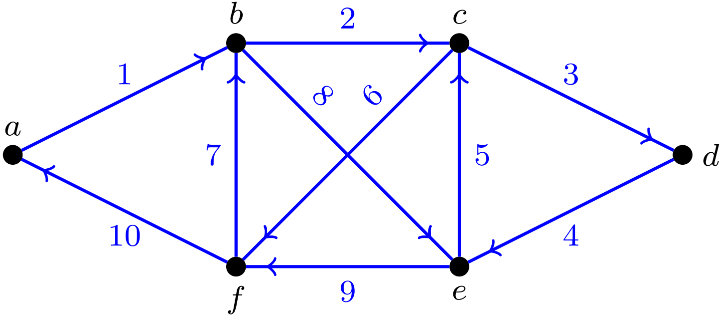
- Step 1: choose vertex a and find a circuit starting at a

Hierholzer's Algorithm



- Step 2: since $\det(b) = 4$, two edges remain from b. We can find a second circuit starting at b.

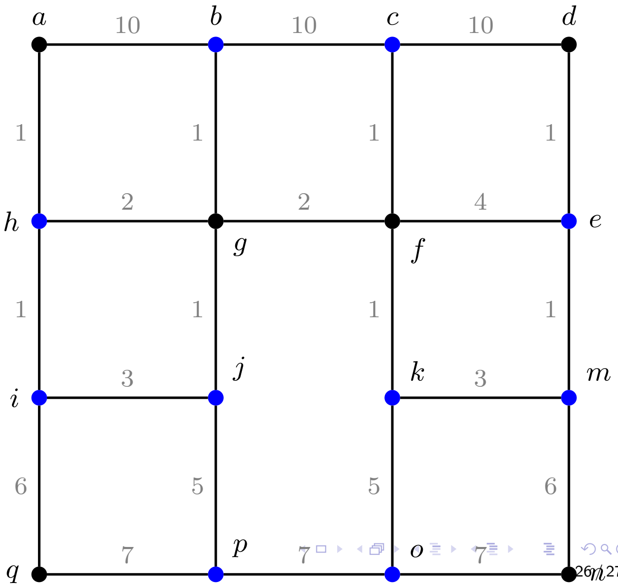
Hierholzer's Algorithm



- Step 3: combine the new circuit with the existing one
- We have obtained an Eulerian circuit

Chinese Postman Problem

- Find an optimal Eulerization, weights given are in terms of cost
- The Eulerization for the unweighted
- We first attempt to pair vertices along paths using edges of weight 1 as much as possible
- We are able to do this for all vertices except o and p, and we duplicate edge op



Chinese Postman Problem

- We first attempt to pair vertices along paths using edges of weight 1 as much as possible
- We are able to do this for all vertices except o and p, and we duplicate edge op
- Any attempt to avoid using this edge would still require both o and p to be paired with another vertex and would require the use of edges ko and jp, both of which have weight 5.
- There is no way to pair the remaining odd vertices while maintaining a total increase in weight of 13.