Algebra and Discrete Mathematics (ADM)

Tutorial 7 Fundamental spaces and decompositions

Lecturer: Bc. Xiaolu Hou, PhD.
xiaolu.hou@stuba.sk

Elementary row operations and row space

Theorem

Elementary row operations do not change the row space of a matrix.

- It is clear that swapping rows does not affect the row space
- If we multiply a row by a nonzero scalar, we replace a row vector with its scalar multiple, every vector in the span of the original rows can still be written as a linear combination of the modified rows. Vector spaces are closed under scalar multiplication.
- If we replace R_j with $R_j + \alpha R_i$, then every vector in the span of the original rows can still be written as a linear combination of the modified rows. Vector spaces are closed under addition and scalar multiplication.

Elementary row operations may change column spaces

Consider

$$A = \begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 3 \\ 0 & 0 \end{pmatrix}$$

B can be obtained from A by row operation

$$A \xrightarrow{R_2 \to -2R_1 + R_2} B$$

- Column space of A consists of scalar multiples of $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$
- Column space of B consists of scalar multiples of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Row space and column space of matrix in reduced row echelon form

Theorem

If a matrix R is in row echelon form, then the row vectors with the leading 1's (the nonzero row vectors) form a basis for the row space of R, and the pivot columns form a basis for the column space of R.

- Each nonzero row has a leading 1 in a unique column, which is not shared by any lower row. This ensures that no row can be written as a linear combination of the others $\Longrightarrow$ the nonzero rows are linearly independent.
- Each pivot column contains a leading 1 and zeros below it, no pivot column can be written as a linear combination of the others $\Longrightarrow$ the pivot columns are linearly independent.
- Every non-pivot column is a linear combination of the previous pivot columns

Row space and column space of matrix in reduced row echelon form

Every non-pivot column is a linear combination of the previous pivot columns

- Let $\boldsymbol{b}_1, \boldsymbol{b}_2 \dots, \boldsymbol{b}_r$ be the pivot columns before a non-pivot column $\boldsymbol{b}$
- The entries starting from row $r+1$ are all zero for $\boldsymbol{b}_1, \boldsymbol{b}_2 \dots, \boldsymbol{b}_r$ and $\boldsymbol{b}$
- Consider the first r entries of the vectors $\boldsymbol{b}_1, \boldsymbol{b}_2 \dots, \boldsymbol{b}_r$ and $\boldsymbol{b}$, denoted as $\boldsymbol{a}_1, \boldsymbol{a}_2, \dots, \boldsymbol{a}_r$ and $\boldsymbol{a}$ respectively
- The coefficient matrix of the linear system

$$\alpha_1 \boldsymbol{a}_1 + \alpha_2 \boldsymbol{a}_2 + \cdots + \alpha_r \boldsymbol{a}_r = \boldsymbol{a}$$

  is an upper triangular matrix with all main diagonal entries $= 1 \implies$ the system is consistent.

Elementary row operations and column spaces

Theorem

Elementary row operations do not alter dependence relationships or linear independence among the column vectors

- Suppose columns $\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_r$ are linearly dependent, then the linear system

$$\alpha_1 \boldsymbol{w}_1 + \alpha_2 \boldsymbol{w}_2 + \cdots \alpha_r \boldsymbol{w}_r = \mathbf{0}, \tag{1}$$

  has non-trivial solutions.
- Suppose after one elementary operation, those columns become $\boldsymbol{w}'_1, \boldsymbol{w}'_2, \ldots, \boldsymbol{w}'_r$
- The coefficient matrix of the linear system

$$\alpha_1 \boldsymbol{w}'_1 + \alpha_2 \boldsymbol{w}'_2 + \cdots \alpha_r \boldsymbol{w}'_r = \mathbf{0},$$

  is obtained from the coefficient matrix of Equation 1 through elementary row operations, they share the same solutions

Linearly independent eigenvectors

Theorem

If $\lambda_1, \lambda_2, \ldots, \lambda_k$ are distinct eigenvalues of A, and if $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k$ are corresponding eigenvectors, then $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k\}$ is a linearly independent set.

Proof.

- Assume $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k$ are linearly dependent. Since an eigenvector is by definition nonzero, $\{\boldsymbol{v}_1\}$ is linearly independent.
- Let r be the largest integer s.t. $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_r$ is linearly independent.
- Then $1 \leq r < k$
- $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_{r+1}$ is linearly dependent

$$\alpha_1 \boldsymbol{v}_1 + \alpha_2 \boldsymbol{v}_2 + \cdots + \alpha_{r+1} \boldsymbol{v}_{r+1} = \mathbf{0},$$

  where α_i's are not all zero
- Multiplying both sides by A and use the fact that $A\boldsymbol{v}_i = \lambda_i \boldsymbol{v}_i$, we get

Linearly independent eigenvectors

Proof.

$$\alpha_1 \boldsymbol{v}_1 + \alpha_2 \boldsymbol{v}_2 + \cdots + \alpha_{r+1} \boldsymbol{v}_{r+1} = \mathbf{0}, \tag{2}$$

$$\alpha_1 \lambda_1 \boldsymbol{v}_1 + \alpha_2 \lambda_2 \boldsymbol{v}_2 + \cdots + \alpha_{r+1} \lambda_{r+1} \boldsymbol{v}_{r+1} = \mathbf{0} \tag{3}$$

- Subtract Eq (2)$\times \lambda_{r+1}$ from Eq (3)

$$\alpha_1(\lambda_1 - \lambda_{r+1})\boldsymbol{v}_1 + \alpha_2(\lambda_2 - \lambda_{r+1})\boldsymbol{v}_2 + \cdots + \alpha_r(\lambda_r - \lambda_{r+1})\boldsymbol{v}_r = \mathbf{0}$$

- Since $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_r$ is linearly independent

$$\alpha_1(\lambda_1 - \lambda_{r+1}) = \alpha_2(\lambda_2 - \lambda_{r+1}) = \cdots = \alpha_r(\lambda_r - \lambda_{r+1}) = 0$$

λ_i's are distinct, we have

$$c_1 = c_2 = \cdots = c_r = 0$$

Substituting back to Eq (2) gives $\alpha_{r+1}\boldsymbol{v}_{r+1} = \mathbf{0}$. Since $\boldsymbol{v}_{r+1} \neq \mathbf{0}$, it follows that $\alpha_{r+1} = 0$, a contradiction to the choice of r.

LU-decomposition – Doolittle decomposition

$$A = \begin{pmatrix} 2 & -1 \\ 3 & 3 \end{pmatrix}, \quad L = \begin{pmatrix} 1 & 0 \\ \ell_{21} & 1 \end{pmatrix}, \quad U = \begin{pmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{pmatrix}$$

- Row 1

$$u_{11} = 2, \quad u_{12} = -1$$

- Row 2

$$\ell_{21} u_{11} = 3 \Longrightarrow \ell_{21} = \frac{3}{2}$$

$$\ell_{21} u_{12} + u_{22} = 3 \Longrightarrow u_{22} = 3 - \frac{3}{2} \times (-1) = \frac{9}{2}$$

- We have

$$L = \begin{pmatrix} 1 & 0 \\ \frac{3}{2} & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 2 & -1 \\ 0 & \frac{9}{2} \end{pmatrix}$$

Solving linear system

Consider the linear system $A\boldsymbol{x} = \boldsymbol{b}$, where

$$A = \begin{pmatrix} 2 & -1 \\ 3 & 3 \end{pmatrix}, \quad \boldsymbol{b} = \begin{pmatrix} 0 \\ 9 \end{pmatrix}$$

We have computed $A = LU$, we have $L\boldsymbol{y} = \boldsymbol{b}$

$$\begin{pmatrix} 1 & 0 \\ \frac{3}{2} & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 9 \end{pmatrix} \Longrightarrow y_1 = 0, \ y_2 = 9$$

$U\boldsymbol{x} = \boldsymbol{y}$

$$\begin{pmatrix} 2 & -1 \\ 0 & \frac{9}{2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 9 \end{pmatrix} \Longrightarrow x_2 = 2, \ x_1 = 1$$

Eigenvalues and eigenvectors

$$A = \begin{pmatrix} 5 & 4 \\ 8 & 9 \end{pmatrix}$$

Characteristic equation of A is

$$\det(\lambda I) = \begin{vmatrix} \lambda - 5 & -4 \\ -8 & \lambda - 9 \end{vmatrix} = (\lambda - 5)(\lambda - 9) - 32$$

Solving for λ

$$\lambda^2 - 14\lambda + 13 = 0 \implies (\lambda - 13)(\lambda - 1) = 0 \implies \lambda_1 = 1,\ \lambda_2 = 13$$

Eigenvalues and eigenvectors

$$A = \begin{pmatrix} 5 & 4 \\ 8 & 9 \end{pmatrix}, \quad \lambda_1 = 1,\ \lambda_2 = 13$$

$$\begin{pmatrix} \lambda - 5 & -4 \\ -8 & \lambda - 9 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \xrightarrow{\lambda_1 = 1} \begin{pmatrix} -4 & -4 \\ -8 & -8 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = t \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

A basis for the eigenspace corresponding to eigenvalue $\lambda_1 = 1$ is $\{\begin{pmatrix} -1 & 1 \end{pmatrix}\}$

$$\begin{pmatrix} \lambda - 5 & -4 \\ -8 & \lambda - 9 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \xrightarrow{\lambda_2 = 13} \begin{pmatrix} 8 & -4 \\ -8 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Longrightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = t \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

A basis for the eigenspace corresponding to eigenvalue $\lambda_2 = 13$ is $\{\begin{pmatrix} 2 & 1 \end{pmatrix}\}$