Algebra and Discrete Mathematics (ADM)

Tutorial 6 Matrix operators

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Matrix transformations

- Consider $T : \mathbb{R}^2 \to \mathbb{R}^3$ with standard matrix

$$[T] = \begin{pmatrix} -1 & -1 \\ 2 & 3 \\ 3 & 1 \end{pmatrix}$$

- Find the image of $\boldsymbol{x} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$

$$T(\boldsymbol{x}) = \begin{pmatrix} -1 & -1 \\ 2 & 3 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 4 \end{pmatrix} = \begin{pmatrix} -5 \\ 14 \\ 7 \end{pmatrix}$$

Matrix transformations

- Consider $T : \mathbb{R}^3 \to \mathbb{R}^2$ with standard matrix

$$[T] = \begin{pmatrix} -2 & -1 & 0 \\ 1 & 2 & 3 \end{pmatrix}$$

- Find the image of $\boldsymbol{x} = \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix}$

$$T(\boldsymbol{x}) = \begin{pmatrix} -2 & -1 & 0 \\ 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} -11 \\ 16 \end{pmatrix}$$

Reflection operators on $\mathbb{R}^2$

- Reflection about the $x-$axis, $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
- Reflection about the $y-$axis, $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$
- Reflection about the line $y = x$, $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
- Reflection about the line $y = \sqrt{3}x$
  - $y = \sqrt{3}x$ makes an angle $\pi/3$ $(= 60^\circ)$ with positive $x-$axis

$$\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix} = \begin{pmatrix} -\dfrac{1}{2} & \dfrac{\sqrt{3}}{2} \\ \dfrac{\sqrt{3}}{2} & \dfrac{1}{2} \end{pmatrix}$$

Reflection operators on $\mathbb{R}^3$

- Reflection about the $xy-$plane

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

- Reflection about the $xz-$plane

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- Reflection about the $yz-$plane

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Projection operators on $\mathbb{R}^2$

- Orthogonal projection onto the $x-$axis

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

- Orthogonal projection onto the $y-$axis

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Projection operators on $\mathbb{R}^3$

- Orthogonal projection onto the $xy-$plane

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

- Orthogonal projection onto the $xz-$plane

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- Orthogonal projection onto the $yz-$plane

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Rotation operators on $\mathbb{R}^2$

- Moves points *counterclockwise* about the origin through a positive angle θ
- *Rotation matrix*

$$R_\theta := \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

- Clockwise about the origin through an angle θ

$$R_{-\theta} := \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

Rotation operators on $\mathbb{R}^3$

Operator	Rotation equations	Standard matrix
Counterclockwise rotation about the positive $x-$axis through an angle θ	$w_1 = x$ $w_2 = y\cos\theta - z\sin\theta$ $w_3 = y\sin\theta + z\cos\theta$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$
Counterclockwise rotation about the positive $y-$axis through an angle θ	$w_1 = x\cos\theta + z\sin\theta$ $w_2 = y$ $w_3 = -x\sin\theta + z\cos\theta$	$\begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$
Counterclockwise rotation about the positive $z-$axis through an angle θ	$w_1 = x\cos\theta - y\sin\theta$ $w_2 = x\sin\theta + y\cos\theta$ $w_3 = z$	$\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Dilations and contractions

- Dilation/contraction with factor α on $\mathbb{R}^3$, $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha x \\ \alpha y \end{pmatrix}$

$$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}$$

- Dilation/contraction with factor α on $\mathbb{R}^3$, $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \alpha x \\ \alpha y \\ \alpha z \end{pmatrix}$

$$\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{pmatrix}$$

Expansions and compressions on $\mathbb{R}^2$

- In the $x-$direction – $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha x \\ y \end{pmatrix}$

$$\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}$$

- In the $y-$direction – $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ \alpha y \end{pmatrix}$

$$\begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}$$

Shears on $\mathbb{R}^2$

- Shear in the $x-$direction by a factor α, $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + \alpha y \\ y \end{pmatrix}$

$$\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$$

- Shear in the $y-$direction by a factor α, $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y + \alpha x \end{pmatrix}$

$$\begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix}$$

Composition of matrix transformations

- Consider a square $ABCD$ with vertices

$$A = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \; B = \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \; C = \begin{pmatrix} 3 \\ 3 \end{pmatrix}, \; D = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

- Perform the following transformations
  - T_1: shear in the $x-$direction by a factor 2
  - T_2: reflection about the $x-$axis
  - T_3: reflection about the $y-$axis

$$[T_1] = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \quad [T_2] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad [T_3] = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$[T_3][T_2][T_1] \begin{pmatrix} 1 & 3 & 3 & 1 \\ 1 & 1 & 3 & 3 \end{pmatrix} = \begin{pmatrix} -3 & -5 & -9 & -7 \\ -1 & -1 & -3 & -3 \end{pmatrix}$$