Algebra and Discrete Mathematics (ADM)

Tutorial 5 Vector spaces and linear independence

Lecturer: Bc. Xiaolu Hou, PhD.
xiaolu.hou@stuba.sk

Vector space of polynomials

- Let $\mathbb{R}[x]$ denote the set of polynomials with coefficients from $\mathbb{R}$, i.e.

$$\mathbb{R}[x] = \left\{ \sum_{i=0}^{n} a_i x^i \;\middle|\; a_i \in \mathbb{R}, n \geq 0 \right\}$$

- An element of $\mathbb{R}[x]$ is of the form $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \in \mathbb{R}[x]$
- If $a_n \neq 0$, we define *degree of* $f(x)$, denoted $\deg(f(x))$, to be n.
- Following the convention, we define $\deg(0) = -\infty$.

Vector space of polynomials

$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$,
$g(x) = b_m x^m + b_{m-1} x^{m-1} + \cdots + b_0$ in $\mathbb{R}[x]$
Without loss of generality, let us assume $n \geq m$, write

$$g(x) = b_n x^n + b_{n-1} x^{n-1} + \cdots + b_0,$$

where $b_i = 0$ for $i > m$. Then

$$f(x) + g(x) := c_n x^n + c_{n-1} x^{n-1} + \cdots + c_0, \text{ where } c_i = a_i + b_i.$$

And

$$f(x)g(x) := d_n x^n + d_{n-1} x^{n-1} + \cdots + d_0, \text{ where } d_i = \sum_{j=0}^{i} a_j b_{i-j}.$$

$(\mathbb{R}[x], +, \cdot)$ is a vector space

Vector space of polynomials

$$\mathbb{R}[x] = \left\{ \sum_{i=0}^{n} a_i x^i \;\middle|\; a_i \in \mathbb{R}, n \geq 0 \right\}$$

- $(\mathbb{R}[x], +, \cdot)$ is a vector space
- Additive identity/zero vector: 0
- Additive inverse of $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$ is

$$-a_n x^n - a_{n-1} x^{n-1} - \cdots - a_0$$

- Other axioms also hold, following from the properties of real numbers

Linear independent vectors

Given the vectors:

$$\boldsymbol{u} = (-1, \quad 2, \quad -3), \quad \boldsymbol{v} = (1, \quad 0, \quad 4), \quad \boldsymbol{w} = (2, \quad -2, \quad 1)$$

Are these vectors linearly dependent or independent?
We compute the determinant:

$$\begin{vmatrix} -1 & 1 & 2 \\ 2 & 0 & -2 \\ -3 & 4 & 1 \end{vmatrix}$$

Apply Sarrus' rule

$$0 + 16 + 6 - (0 + 8 + 2) = 12 \neq 0$$

Since the determinant is nonzero, the vectors are **linearly independent**.

Linear independent vectors

Given the vectors:

$$\boldsymbol{u} = (2, \quad 3)\,, \quad \boldsymbol{v} = (5, \quad -1)\,, \quad \boldsymbol{w} = (1, \quad 4)$$

Are these vectors linearly dependent?
Consider

$$\alpha_1\,(2, \quad 3) + \alpha_2\,(5, \quad -1) + \alpha_3\,(1, \quad 4) = (0, \quad 0)\,,$$

which corresponds to the following homogeneous linear system

$$\begin{aligned} 2\alpha_1 + 5\alpha_2 + \alpha_3 &= 0 \\ 3\alpha_1 - \alpha_2 + 4\alpha_3 &= 0 \end{aligned}$$

Recall

Corollary

A homogeneous linear system with more unknowns than equations has infinitely many solutions.

The system has nontrivial solutions, the vectors are **linearly dependent**.

Linear independent vectors

Are the following vectors linearly independent?

$$\boldsymbol{v}_1 = \begin{pmatrix}1, & 3, & 5, & 0\end{pmatrix}, \quad \boldsymbol{v}_2 = \begin{pmatrix}-1, & 2, & 1, & 1\end{pmatrix},$$

$$\boldsymbol{v}_3 = \begin{pmatrix}2, & 2, & 0, & -1\end{pmatrix}, \quad \boldsymbol{v}_4 = \begin{pmatrix}1, & 0, & 1, & 0\end{pmatrix}$$

Cofactor expansion along the fourth row, and Sarru's rule

$$\begin{aligned}
\begin{vmatrix}1 & -1 & 2 & 1\\ 3 & 2 & 2 & 0\\ 5 & 1 & 0 & 1\\ 0 & 1 & -1 & 0\end{vmatrix} &= (-1)^{4+2}\begin{vmatrix}1 & 2 & 1\\ 3 & 2 & 0\\ 5 & 0 & 1\end{vmatrix} - (-1)^{4+3}\begin{vmatrix}1 & -1 & 1\\ 3 & 2 & 0\\ 5 & 1 & 1\end{vmatrix}\\
&= (2-10-6)-(2+3-10+3)\\
&= -14-2=-16
\end{aligned}$$

Linear combination

$$\boldsymbol{u} = \begin{pmatrix}1, & 2, & 3\end{pmatrix}, \quad \boldsymbol{v} = \begin{pmatrix}3, & 2, & 1\end{pmatrix}, \quad \boldsymbol{w} = \begin{pmatrix}6, & 8, & 10\end{pmatrix}$$

Is $\boldsymbol{w}$ a linear combination of $\boldsymbol{u}$ and $\boldsymbol{v}$

$$\begin{vmatrix}1 & 3 & 6\\ 2 & 2 & 8\\ 3 & 1 & 10\end{vmatrix} = 0 \Longrightarrow \text{the three vectors are linearly dependent}$$

$$\alpha_1 \boldsymbol{u} + \alpha_2 \boldsymbol{v} = \boldsymbol{w}$$

$$\begin{aligned}\alpha_1 + 3\alpha_2 &= 6\\ 2\alpha_1 + 2\alpha_2 &= 8\\ 3\alpha_1 + \alpha_2 &= 10\end{aligned}$$

Solve by, e.g. Cramer's rule

$$\alpha_1 = 3, \quad \alpha_2 = 1 \Longrightarrow \boldsymbol{w} = 3\boldsymbol{u} + \boldsymbol{v}$$

Linearly independent matrices

Are the three matrices linearly independent in $\mathcal{M}_{2\times 2}$?

$$M_1 = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}$$

$$\alpha_1 M_1 + \alpha_2 M_2 + \alpha_3 M_3 = O$$

gives

$$\begin{pmatrix} \alpha_1 & 0 \\ \alpha_1 & 2\alpha_1 \end{pmatrix} + \begin{pmatrix} \alpha_2 & 2\alpha_2 \\ 2\alpha_2 & \alpha_2 \end{pmatrix} + \begin{pmatrix} 0 & \alpha_3 \\ 2\alpha_3 & \alpha_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\begin{aligned} \alpha_1 + \alpha_2 &= 0 \\ 2\alpha_2 + \alpha_3 &= 0 \\ \alpha_1 + 2\alpha_2 + 2\alpha_3 &= 0 \\ 2\alpha_1 + \alpha_2 + \alpha_3 &= 0 \end{aligned}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 1 & 2 & 2 & 0 \\ 2 & 1 & 1 & 0 \end{array}\right)$$

Linearly independent matrices

Are the three matrices linearly independent in $\mathcal{M}_{2\times 2}$?

$$M_1 = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 1 & 2 & 2 & 0 \\ 2 & 1 & 1 & 0 \end{array}\right) \xrightarrow[R_3 \to R_3 - R_1]{R_4 \to R_4 - 2R_1} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & -1 & 1 & 0 \end{pmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & -1 & 1 & 0 \end{pmatrix} \xrightarrow[R_3 \to R_3 - 2R_2]{R_4 \to R_4 + R_2}$$

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 3 & 0 \end{pmatrix} \xrightarrow{R_4 \to R_4 + R_3} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \implies \text{only trivial solution}$$

$\implies$ the matrices are linearly independent