Algebra and Discrete Mathematics (ADM)

Tutorial 4 Determinants

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Determinant of 2×2 matrices

$$A = \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix},$$

$$\det(A) = 2 \times 1 - 3 \times 0 = 2$$

Cofactor expansion

$$A = \begin{pmatrix} 4 & 2 & -1 \\ 0 & 2 & -3 \\ -1 & 1 & 5 \end{pmatrix}$$

Cofactor expansion along the second row

$$\begin{aligned} \det(A) &= 0 + (-1)^{2+2} \times 2 \begin{vmatrix} 4 & -1 \\ -1 & 5 \end{vmatrix} + (-1)^{2+3} \times (-3) \begin{vmatrix} 4 & 2 \\ -1 & 1 \end{vmatrix} \\ &= 2 \times (20 - 1) + 3 \times (4 + 2) = 38 + 18 = 56 \end{aligned}$$

Cofactor expansion

$$A = \begin{pmatrix} 1 & 3 & 1 & 5 \\ -2 & -7 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \end{pmatrix}$$

Cofactor expansion along the third row

$$\det(A) = 0$$

Row operations and determinant

$$A = \begin{pmatrix} 3 & 3 & 0 & 5 \\ 2 & 2 & 0 & -2 \\ 4 & 1 & -3 & 0 \\ 2 & 10 & 3 & 2 \end{pmatrix}$$

$$|A| \overset{R_1 \to R_1 - R_2}{=\!=} \begin{vmatrix} 1 & 1 & 0 & 7 \\ 2 & 2 & 0 & -2 \\ 4 & 1 & -3 & 0 \\ 2 & 10 & 3 & 2 \end{vmatrix} \overset{R_2 \to R_2 - 2R_1}{=\!=} \begin{vmatrix} 1 & 1 & 0 & 7 \\ 0 & 0 & 0 & -16 \\ 4 & 1 & -3 & 0 \\ 2 & 10 & 3 & 2 \end{vmatrix} = (-16) \times \begin{vmatrix} 1 & 1 & 0 \\ 4 & 1 & -3 \\ 2 & 10 & 3 \end{vmatrix}$$

$$\overset{R_2 \to R_2 - 4R_1}{=\!=} (-16) \times \begin{vmatrix} 1 & 1 & 0 \\ 0 & -3 & -3 \\ 2 & 10 & 3 \end{vmatrix} = 48 \times \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 2 & 10 & 3 \end{vmatrix} \overset{R_3 \to R_3 - 2R_1}{=\!=} 48 \times \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 8 & 3 \end{vmatrix}$$

$$\overset{R_3 \to R_3 - 8R_2}{=\!=} 48 \times \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -5 \end{vmatrix} = 48 \times (-5) = -240$$

Inverse of a matrix using adjugate

$$A = \begin{pmatrix} 3 & 0 & 0 \\ 2 & -1 & 5 \\ 1 & 9 & -4 \end{pmatrix}$$

adjugate matrix

$$\operatorname{adj}(A) = \begin{pmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{pmatrix} = \begin{pmatrix} -41 & 0 & 0 \\ 13 & -12 & -15 \\ 19 & -27 & -3 \end{pmatrix}$$

$$C_{11} = (-1)^{1+1} \times \begin{vmatrix} -1 & 5 \\ 9 & -4 \end{vmatrix} = 4 - 45 = -41, \quad C_{12} = (-1)^{1+2} \begin{vmatrix} 2 & 5 \\ 1 & -4 \end{vmatrix} = 13$$

$$C_{13} = (-1)^{1+3} \begin{vmatrix} 2 & -1 \\ 1 & 9 \end{vmatrix} = 19, \quad \ldots$$

Inverse of a matrix using adjugate

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adjugate matrix

$$\operatorname{adj}(A) = \begin{pmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{pmatrix} = \begin{pmatrix} -41 & 0 & 0 \\ 13 & -12 & -15 \\ 19 & -27 & -3 \end{pmatrix}$$

$$|A| = 3C_{11} = 3 \times (-41) = -123.$$

$$A^{-1} = -\frac{1}{123} \begin{pmatrix} -41 & 0 & 0 \\ 13 & -12 & -15 \\ 19 & -27 & -3 \end{pmatrix}$$

Determinant with unknown constant

$$A = \begin{pmatrix} \lambda - 2 & 1 \\ -5 & \lambda + 4 \end{pmatrix}$$

For what values of λ is A invertible?

$$\det(A) = (\lambda - 2)(\lambda + 4) + 5 = \lambda^2 + 2\lambda - 3$$

$$\det(A) = 0 \iff \lambda = 1, 3$$

A is invertible if $\lambda \neq 1, 3$

$|A+B| = |A| + |B|?$

The equality does not hold, e.g.

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \quad A + B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

$$|A| = 2^3 = 8, \quad |B| = -3, \quad |A+B| = 15$$

Cramer's rule

$$\begin{aligned} 2x + 5y &= 4 \\ 3x - y &= 2 \end{aligned}$$

$$|A| = \begin{vmatrix} 2 & 5 \\ 2 & -1 \end{vmatrix} = -17, \quad |A_1| = \begin{vmatrix} 4 & 5 \\ 2 & -1 \end{vmatrix} = -14, \quad |A_2| = \begin{vmatrix} 2 & 4 \\ 3 & 2 \end{vmatrix} = -8$$

$$x = \frac{|A_1|}{|A|} = \frac{14}{17}, \quad y = \frac{|A_2|}{|A|} = \frac{8}{17}$$

Cramer's rule

$$\begin{aligned} x_1 + 2x_2 + 2x_3 &= 2 \\ -3x_2 - 3x_3 &= 0 \\ 2x_1 + 2x_3 &= 2 \end{aligned}$$

$$|A| = \begin{vmatrix} 1 & 2 & 2 \\ 0 & -3 & -3 \\ 2 & 0 & 2 \end{vmatrix} = -6, \quad |A_1| = \begin{vmatrix} 2 & 2 & \\ 0 & -3 & -3 \\ 2 & 0 & 2 \end{vmatrix} = -12,$$

$$|A_2| = \begin{vmatrix} 1 & 2 & 2 \\ 0 & 0 & -3 \\ 2 & 2 & 2 \end{vmatrix} = -6, \quad |A_3| = \begin{vmatrix} 1 & 2 & 2 \\ 0 & -3 & 0 \\ 2 & 0 & 2 \end{vmatrix} = 6$$

$$x_1 = \frac{|A_1|}{|A|} = \frac{-12}{-6} = 2, \quad x_2 = \frac{|A_2|}{|A|} = \frac{-6}{-6} = 1, \quad x_3 = \frac{|A_3|}{|A|} = \frac{6}{= 6} = -1$$