Algebra and Discrete Mathematics (ADM)

Tutorial 3 Matrix inverse and solving linear systems

Lecturer: Bc. Xiaolu Hou, PhD.
xiaolu.hou@stuba.sk

Inverse of 2×2 matrices

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

when

$$\det(A) = ad - bc \neq 0,$$

we have

$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Inverse of 2×2 matrices

$$A = \begin{pmatrix} 2 & 2 \\ 1 & -1 \end{pmatrix}$$

The determinant of A is

$$\det(A) = 2 \times (-1) - 1 \times 2 = -4 \neq 0$$

Then

$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} -1 & -2 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} \frac{1}{4} & \frac{1}{2} \\ \frac{1}{4} & -\frac{1}{2} \end{pmatrix}$$

Or by row operations

$$\left(\begin{array}{cc|cc} 2 & 2 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{array}\right) \xrightarrow{R_1 \to \frac{1}{2} R_1} \left(\begin{array}{cc|cc} 1 & 1 & \frac{1}{2} & 0 \\ 1 & -1 & 0 & 1 \end{array}\right) \xrightarrow{R_2 \to R_2 - R_1} \left(\begin{array}{cc|cc} 1 & 1 & \frac{1}{2} & 0 \\ 0 & -2 & -\frac{1}{2} & 1 \end{array}\right)$$

Inverse of 2×2 matrices

$$A = \begin{pmatrix} 2 & 2 \\ 1 & -1 \end{pmatrix}$$

By row operations

$$\left(\begin{array}{cc|cc} 1 & 1 & \frac{1}{2} & 0 \\ 0 & -2 & -\frac{1}{2} & 1 \end{array}\right) \xrightarrow{R_2 \to -\frac{1}{2} R_2} \left(\begin{array}{cc|cc} 1 & 1 & \frac{1}{2} & 0 \\ 0 & 1 & \frac{1}{4} & -\frac{1}{2} \end{array}\right) \xrightarrow{R_1 \to R_1 - R_2} \left(\begin{array}{cc|cc} 1 & 0 & \frac{1}{4} & \frac{1}{2} \\ 0 & 1 & \frac{1}{4} & -\frac{1}{2} \end{array}\right)$$

Inverse of 2×2 matrices

For $A \in \mathcal{M}_{2\times 2}$, when do we have $A^{-1} = A$?

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$a = \frac{d}{\det(A)}, \quad b = -\frac{b}{\det(A)}, \quad c = -\frac{c}{\det(A)}, \quad d = \frac{a}{\det(A)}$$

- $b = c = 0$, $a = d$, $\det(A) = a^2 = 1$

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad -I_2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

- $\det(A) = -1$, $a = -d$

$$A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}, \quad a^2 + bc = 1$$

Inverse of 2×2 matrices

$$A = \begin{pmatrix} 4 & 3 \\ -5 & -4 \end{pmatrix}$$

Then

$$A^{-1} = \frac{1}{4 \times (-4) - 3 \times (-5)} \begin{pmatrix} -4 & -3 \\ 5 & 4 \end{pmatrix} = - \begin{pmatrix} -4 & -3 \\ 5 & 4 \end{pmatrix} = \begin{pmatrix} 4 & 3 \\ -5 & -4 \end{pmatrix} = A$$

Solve linear system using matrix inverse

$$\begin{aligned} -y + z &= 2 \\ 2x + 3y &= 12 \\ -x - z &= -7 \end{aligned}$$

The system can be presented by $A\boldsymbol{x} = \boldsymbol{b}$, where

$$A = \begin{pmatrix} 0 & -1 & 1 \\ 2 & 3 & 0 \\ -1 & 0 & -1 \end{pmatrix}, \quad \boldsymbol{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad \boldsymbol{b} = \begin{pmatrix} 2 \\ 12 \\ -7 \end{pmatrix}$$

Find inverse of A

$$\left(\begin{array}{ccc|ccc} 0 & -1 & 1 & 1 & 0 & 0 \\ 2 & 3 & 0 & 0 & 1 & 0 \\ -1 & 0 & -1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{(-R_3)\leftrightarrow R_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & -1 \\ 2 & 3 & 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & 1 & 0 & 0 \end{array}\right)$$

Solve linear system using matrix inverse

$$\begin{aligned} -y + z &= 2 \\ 2x + 3y &= 12 \\ -x - z &= -7 \end{aligned}$$

Find inverse of A

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & -1 \\ 2 & 3 & 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & 1 & 0 & 0 \end{array}\right) \xrightarrow{R_2 \to R_2 - 2R_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & -1 \\ 0 & 3 & -2 & 0 & 1 & 2 \\ 0 & -1 & 1 & 1 & 0 & 0 \end{array}\right) \xrightarrow[R_2 \leftrightarrow (-R_3)]{R_2 \to R_2 + 3R_3}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 & 2 \end{array}\right) \xrightarrow[R_1 \to R_1 - R_3]{R_2 \to R_2 + R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -3 & -1 & -3 \\ 0 & 1 & 0 & 2 & 1 & 2 \\ 0 & 0 & 1 & 3 & 1 & 2 \end{array}\right)$$

Solve linear system using matrix inverse

$$\begin{aligned} -y + z &= 2 \\ 2x + 3y &= 12 \\ -x - z &= -7 \end{aligned}$$

$$A^{-1} = \begin{pmatrix} -3 & -1 & -3 \\ 2 & 1 & 2 \\ 3 & 1 & 2 \end{pmatrix}$$

The solution is given by

$$\boldsymbol{x} = A^{-1}\boldsymbol{b} = \begin{pmatrix} -3 & -1 & -3 \\ 2 & 1 & 2 \\ 3 & 1 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ 12 \\ -7 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix}$$

Solve linear system using matrix inverse

$$A = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 2 & 1 & 1 & 1 \\ 1 & 1 & 0 & -1 \\ 1 & 2 & 2 & 0 \end{pmatrix}, \quad \boldsymbol{b}_1 = \begin{pmatrix} 2 \\ 4 \\ -2 \\ 7 \end{pmatrix}, \quad \boldsymbol{b}_2 = \begin{pmatrix} 4 \\ 8 \\ -7 \\ 8 \end{pmatrix}$$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 2 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & -1 & 0 & 0 & 1 & 0 \\ 1 & 2 & 2 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \xrightarrow[R_3 \to R_3 - R_1,\ R_4 \to R_4 - R_1]{R_2 \to R_2 - 2R_1} \left(\begin{array}{cccc|cccc} 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & -1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -2 & -1 & 0 & 1 & 0 \\ 0 & 2 & 1 & -1 & -1 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow[R_4 \to R_4 - 2R_2]{R_3 \to R_3 - R_2} \left(\begin{array}{cccc|cccc} 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & -1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 3 & 1 & 3 & -2 & 0 & 1 \end{array}\right)$$

Solve linear system using matrix inverse

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & -1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 3 & 1 & 3 & -2 & 0 & 1 \end{array}\right) \xrightarrow{\frac{1}{3}R_4 \leftrightarrow (-R_3)} \left(\begin{array}{cccc|cccc} 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & -1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{1}{3} & 1 & -\frac{2}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 1 & -1 & 1 & -1 & 0 \end{array}\right)$$

$$\xrightarrow[R_1 \to R_1 - R_3]{R_2 \to R_2 + R_3} \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & \frac{2}{3} & 0 & \frac{2}{3} & 0 & -\frac{1}{3} \\ 0 & 1 & 0 & -\frac{2}{3} & -1 & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 1 & \frac{1}{3} & 1 & -\frac{2}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 1 & -1 & 1 & -1 & 0 \end{array}\right)$$

Solve linear system using matrix inverse

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & \frac{2}{3} & 0 & \frac{2}{3} & 0 & -\frac{1}{3} \\ 0 & 1 & 0 & -\frac{2}{3} & -1 & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 1 & \frac{1}{3} & 1 & -\frac{2}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 1 & -1 & 1 & -1 & 0 \end{array}\right) \xrightarrow[R_2\to R_2+\frac{2}{3}R_4,\ R_3\to R_3-\frac{1}{3}R_4]{R_1\to R_1-\frac{2}{3}R_4}$$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{2}{3} & 0 & \frac{2}{3} & -\frac{1}{3} \\ 0 & 1 & 0 & 0 & -\frac{5}{3} & 1 & -\frac{2}{3} & \frac{1}{3} \\ 0 & 0 & 1 & 0 & \frac{4}{3} & -1 & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 1 & -1 & 1 & -1 & 0 \end{array}\right) \Longrightarrow A^{-1} = \frac{1}{3}\begin{pmatrix} 2 & 0 & 2 & -1 \\ -5 & 3 & -2 & 1 \\ 4 & -3 & 1 & 1 \\ -3 & 3 & -3 & 0 \end{pmatrix}$$

Solve linear system using matrix inverse

$$A = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 2 & 1 & 1 & 1 \\ 1 & 1 & 0 & -1 \\ 1 & 2 & 2 & 0 \end{pmatrix}, \quad \boldsymbol{b}_1 = \begin{pmatrix} 2 \\ 4 \\ -2 \\ 7 \end{pmatrix}, \quad \boldsymbol{b}_2 = \begin{pmatrix} 4 \\ 8 \\ -7 \\ 8 \end{pmatrix}$$

$$\boldsymbol{x}_1 = A^{-1}\boldsymbol{b}_1 = \frac{1}{3}\begin{pmatrix} 2 & 0 & 2 & -1 \\ -5 & 3 & -2 & 1 \\ 4 & -3 & 1 & 1 \\ -3 & 3 & -3 & 0 \end{pmatrix}\begin{pmatrix} 2 \\ 4 \\ -2 \\ 7 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} -7 \\ 13 \\ 1 \\ 12 \end{pmatrix}$$

$$\boldsymbol{x}_2 = A^{-1}\boldsymbol{b}_2 = \frac{1}{3}\begin{pmatrix} 2 & 0 & 2 & -1 \\ -5 & 3 & -2 & 1 \\ 4 & -3 & 1 & 1 \\ -3 & 3 & -3 & 0 \end{pmatrix}\begin{pmatrix} 4 \\ 8 \\ -7 \\ 8 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} -14 \\ 26 \\ -7 \\ 33 \end{pmatrix}$$

Solve linear system using matrix inverse

$$A = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 2 & 1 & 1 & 1 \\ 1 & 1 & 0 & -1 \\ 1 & 2 & 2 & 0 \end{pmatrix}, \quad \boldsymbol{b}_1 = \begin{pmatrix} 2 \\ 4 \\ -2 \\ 7 \end{pmatrix}, \quad \boldsymbol{b}_2 = \begin{pmatrix} 4 \\ 8 \\ -7 \\ 8 \end{pmatrix}$$

We can verify that

$$A\boldsymbol{x}_1 = \frac{1}{3} \begin{pmatrix} 1 & 0 & 1 & 1 \\ 2 & 1 & 1 & 1 \\ 1 & 1 & 0 & -1 \\ 1 & 2 & 2 & 0 \end{pmatrix} \begin{pmatrix} -7 \\ 13 \\ 1 \\ 12 \end{pmatrix} = \boldsymbol{b}_1$$

$$A\boldsymbol{x}_2 = \frac{1}{3} \begin{pmatrix} 1 & 0 & 1 & 1 \\ 2 & 1 & 1 & 1 \\ 1 & 1 & 0 & -1 \\ 1 & 2 & 2 & 0 \end{pmatrix} \begin{pmatrix} -14 \\ 26 \\ -7 \\ 33 \end{pmatrix} = \boldsymbol{b}_2$$