Algebra and Discrete Mathematics (ADM)

Tutorial 2 Linear systems and Gauss–Jordan elimination

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Solving system of linear equations

$$\begin{aligned} 2x + y &= 5 \\ x - y &= -2 \end{aligned}$$

By substitution, the second equation implies $x = -2 + y$, substitute to the first gives

$$2(-2 + y) + y = 5 \Longrightarrow 3y = 9 \Longrightarrow y = 3 \Longrightarrow x = 1$$

By elimination, adding two equations gives

$$3x = 3 \Longrightarrow x = 1$$

$$1 - y = -2 \Longrightarrow y = 3$$

Solving system of linear equations

More complicated system

$$
\begin{aligned}
-x + 4y + z &= -5 \\
2x + 2y + z &= 3 \\
x - 2y - 2 &= 3
\end{aligned}
$$

Convert to matrix form

$$
\begin{pmatrix} -1 & 4 & 1 \\ 2 & 2 & 1 \\ 1 & -2 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -5 \\ 3 \\ 3 \end{pmatrix}
$$

Augmented matrix

$$
\begin{pmatrix} -1 & 4 & 1 & -5 \\ 2 & 2 & 1 & 3 \\ 1 & -2 & -1 & 3 \end{pmatrix}
$$

Solving system of linear equations

$$\begin{pmatrix} -1 & 4 & 1 & -5 \\ 2 & 2 & 1 & 3 \\ 1 & -2 & -1 & 3 \end{pmatrix}$$

Elementary row operations for Gauss–Jordan elimination

- Multiply a row by a nonzero constant
- Interchange two rows
- Add a constant times one row to another

$$\xrightarrow{R_1 \to -1 R_1} \begin{pmatrix} 1 & -4 & -1 & 5 \\ 2 & 2 & 1 & 3 \\ 1 & -2 & -1 & 3 \end{pmatrix} \xrightarrow[R_2 \to R_2 - 2R_1]{R_3 \to R_3 - R_1} \begin{pmatrix} 1 & -4 & -1 & 5 \\ 0 & 10 & 3 & -7 \\ 0 & 2 & 0 & -2 \end{pmatrix} \xrightarrow{R_2 \to \frac{1}{10} R_2}$$

$$\begin{pmatrix} 1 & -4 & -1 & 5 \\ 0 & 1 & \frac{3}{10} & -\frac{7}{10} \\ 0 & 2 & 0 & -2 \end{pmatrix} \xrightarrow{R_3 \to R_3 - 2R_2} \begin{pmatrix} 1 & -4 & -1 & 5 \\ 0 & 1 & \frac{3}{10} & -\frac{7}{10} \\ 0 & 0 & -\frac{3}{5} & -\frac{3}{5} \end{pmatrix} \xrightarrow{R_3 \to -\frac{3}{5} R_3} \begin{pmatrix} 1 & -4 & -1 & 5 \\ 0 & 1 & \frac{3}{10} & -\frac{7}{10} \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

We have reached row echelon form

Solving system of linear equations

Backward:

$$\begin{pmatrix} 1 & -4 & -1 & 5 \\ 0 & 1 & \frac{3}{10} & -\frac{7}{10} \\ 0 & 0 & 1 & 1 \end{pmatrix} \xrightarrow[R_1 \to R_1 + R_3]{R_2 \to R_2 - \frac{3}{10} R_3} \begin{pmatrix} 1 & -4 & 0 & 6 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \xrightarrow{R_1 \to R_1 + 4R_2} \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

Solving system of linear equations

$$\begin{aligned} -x + 4y + z &= -5 \\ 2x + 2y + z &= 3 \\ x - 2y - 2 &= 3 \end{aligned}$$

$$\begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

The solution is given by

$$x = 2, \quad y = -1, \quad z = 1$$

Gauss–Jordan elimination

$$\begin{pmatrix} 1 & 2 & 1 & 4 \\ 0 & 1 & 2 & 3 \\ 1 & 0 & -1 & 0 \\ 2 & 2 & 1 & 5 \end{pmatrix}$$

The leftmost nonzero column is the first column, it already has leading 1

$$\xrightarrow[R_4 \to R_4 - 2R_1]{R_3 \to R_3 - R_1} \begin{pmatrix} 1 & 2 & 1 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & -2 & -2 & -4 \\ 0 & -2 & -1 & -3 \end{pmatrix} \xrightarrow[R_3 \to R_3 + 2R_2]{R_4 \to R_4 + 2R_2} \begin{pmatrix} 1 & 2 & 1 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 3 & 3 \end{pmatrix}$$

$$\xrightarrow{R_3 \to \frac{1}{2} R_3} \begin{pmatrix} 1 & 2 & 1 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 3 & 3 \end{pmatrix} \xrightarrow{R_4 \to R_4 - 3R_3} \begin{pmatrix} 1 & 2 & 1 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

We have reached row echelon form with Gaussian elimination

Gauss–Jordan elimination

Backward

$$\begin{pmatrix} 1 & 2 & 1 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_1 \to R_1 - 2R_2} \begin{pmatrix} 1 & 0 & -3 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow[R_1 \to R_1 + 3R_3]{R_2 \to R_2 - 2R_3} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Gauss–Jordan elimination

$$\begin{pmatrix} 1 & 2 & -1 & \alpha \\ 2 & 3 & -2 & \beta \\ -1 & -1 & 1 & \gamma \end{pmatrix} \xrightarrow[R_2 \to R_2 - 2R_1]{R_3 \to R_3 + R_1} \begin{pmatrix} 1 & 2 & -1 & \alpha \\ 0 & -1 & 0 & \beta - 2\alpha \\ 0 & 1 & 0 & \gamma + \alpha \end{pmatrix} \xrightarrow{R_2 \to -R_2}$$

$$\begin{pmatrix} 1 & 2 & -1 & \alpha \\ 0 & 1 & 0 & 2\alpha - \beta \\ 0 & 1 & 0 & \gamma + \alpha \end{pmatrix} \xrightarrow[R_1 \to R_1 - 2R_2]{R_3 \to R_3 - R_2} \begin{pmatrix} 1 & 0 & -1 & -3\alpha + 2\beta \\ 0 & 1 & 0 & 2\alpha - \beta \\ 0 & 0 & 0 & \gamma - \alpha + \beta \end{pmatrix}$$

The corresponding linear system is consistent, only if $\gamma - \alpha + \beta = 0$. In this case

$$x - z = -3\alpha + 2\beta, \quad y = 2\alpha - \beta$$