Algebra and Discrete Mathematics (ADM)

Tutorial 1 Vectors and matrices

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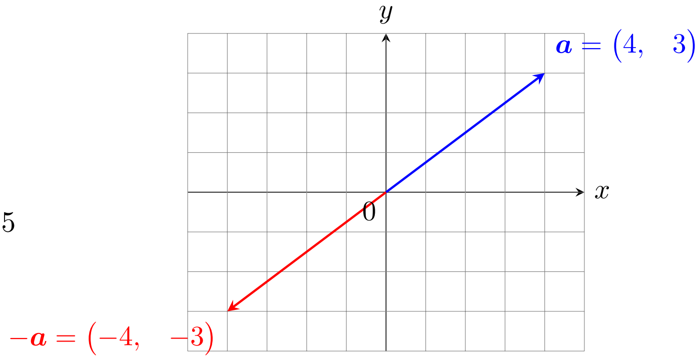
Vector

$\boldsymbol{a} = \begin{pmatrix} 4, & 3 \end{pmatrix},$
$-\boldsymbol{a} = \begin{pmatrix} -4, & -3 \end{pmatrix}$

$\|\boldsymbol{a}\| = \sqrt{4^2 + 3^2} = \sqrt{16 + 9} = 5$



Vector addition and subtraction

$\boldsymbol{a} = \begin{pmatrix} 1, & -3, & 2, & 5 \end{pmatrix}, \boldsymbol{b} = \begin{pmatrix} 2, & 2, & 4, & 0 \end{pmatrix}$

$$\begin{aligned} \boldsymbol{a} + \boldsymbol{b} &= ? \\ \boldsymbol{a} - \boldsymbol{b} &= ? \\ \boldsymbol{b} - \boldsymbol{a} &= ? \end{aligned}$$

Vector addition and subtraction

$$\boldsymbol{a} = \begin{pmatrix}1, & -3, & 2, & 5\end{pmatrix}, \boldsymbol{b} = \begin{pmatrix}2, & 2, & 4, & 0\end{pmatrix}$$

$$\begin{aligned}
\boldsymbol{a}+\boldsymbol{b} &= \boldsymbol{b}+\boldsymbol{a} = \begin{pmatrix}1+2, & -3+2, & 2+4, & 5+0\end{pmatrix} = \begin{pmatrix}3, & -1, & 6, & 5\end{pmatrix} \\
\boldsymbol{a}-\boldsymbol{b} &= \begin{pmatrix}1-2, & -3-2, & 2-4, & 5-0\end{pmatrix} = \begin{pmatrix}-1, & -5, & -2, & 5\end{pmatrix} \\
\boldsymbol{b}-\boldsymbol{a} &= \begin{pmatrix}1, & 5, & 2, & -5\end{pmatrix} = -(\boldsymbol{a}-\boldsymbol{b})
\end{aligned}$$

Projection vectors

- Projection of $\boldsymbol{a}$ onto $\boldsymbol{b}$ is given by

$$\operatorname{proj}_{\boldsymbol{b}} \boldsymbol{a} = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{\|\boldsymbol{b}\|^2} \boldsymbol{b}$$

- Projection of $\boldsymbol{b}$ onto $\boldsymbol{a}$ is given by

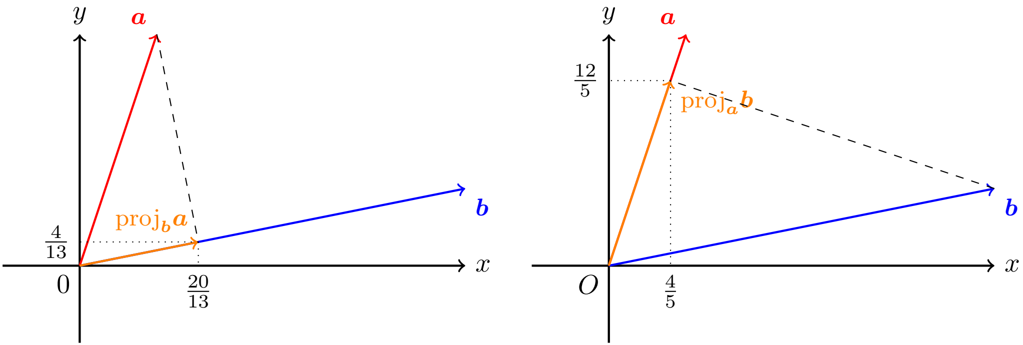
$$\operatorname{proj}_{\boldsymbol{a}} \boldsymbol{b} = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{\|\boldsymbol{a}\|^2} \boldsymbol{a}$$

Example

$\boldsymbol{a} = \begin{pmatrix} 1, & 3 \end{pmatrix}$, $\boldsymbol{b} = \begin{pmatrix} 5, & 1 \end{pmatrix}$

$$\begin{aligned} \operatorname{proj}_{\boldsymbol{b}} \boldsymbol{a} = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{\|\boldsymbol{b}\|^2} \boldsymbol{b} &= \frac{1 \times 5 + 3 \times 1}{5^2 + 1} \boldsymbol{b} = \frac{8}{26} \begin{pmatrix} 5, & 1 \end{pmatrix} = \begin{pmatrix} \frac{20}{13}, & \frac{4}{13} \end{pmatrix} \\ \operatorname{proj}_{\boldsymbol{a}} \boldsymbol{b} = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{\|\boldsymbol{a}\|^2} \boldsymbol{a} &= \frac{1 \times 5 + 3 \times 1}{1^2 + 3^2} \boldsymbol{a} = \frac{8}{10} \begin{pmatrix} 1, & 3 \end{pmatrix} = \begin{pmatrix} \frac{4}{5}, & \frac{12}{5} \end{pmatrix} \end{aligned}$$

Projection vectors



$$\boldsymbol{a} = \begin{pmatrix}1, & 3\end{pmatrix}, \quad \boldsymbol{b} = \begin{pmatrix}5, & 1\end{pmatrix}, \quad \operatorname{proj}_{\boldsymbol{b}} \boldsymbol{a} = \begin{pmatrix}\frac{20}{13}, & \frac{4}{13}\end{pmatrix}, \quad \operatorname{proj}_{\boldsymbol{a}} \boldsymbol{b} = \begin{pmatrix}\frac{4}{5}, & \frac{12}{5}\end{pmatrix}$$

Projection theorem

Theorem

$\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^n$, if $\boldsymbol{a} \neq 0$, then $\boldsymbol{b}$ can be uniquely expressed in the form $\boldsymbol{b} = \boldsymbol{w}_1 + \boldsymbol{w}_2$, where $\boldsymbol{w}_1$ is a scalar multiple of $\boldsymbol{a}$ and $\boldsymbol{w}_2$ is orthogonal to $\boldsymbol{a}$.

Proof.

$$\boldsymbol{b} = \boldsymbol{w}_1 + \boldsymbol{w}_2 = \alpha\boldsymbol{a} + \boldsymbol{w}_2$$

Then

$$\boldsymbol{b} \cdot \boldsymbol{a} = (\alpha\boldsymbol{a} + \boldsymbol{w}_2) \cdot \boldsymbol{a} = \alpha\|\boldsymbol{a}\|^2 + (\boldsymbol{w}_2 \cdot \boldsymbol{a}) = \alpha\|\boldsymbol{a}\|^2 \Longrightarrow \alpha = \frac{\boldsymbol{b} \cdot \boldsymbol{a}}{\|\boldsymbol{a}\|^2}$$

is the only possible value for α.

$$\boldsymbol{b} = \frac{\boldsymbol{b} \cdot \boldsymbol{a}}{\|\boldsymbol{a}\|^2}\boldsymbol{a} + \boldsymbol{w}_2 = \operatorname{proj}_{\boldsymbol{a}} \boldsymbol{b} + \boldsymbol{w}_2$$

□

Projection theorem

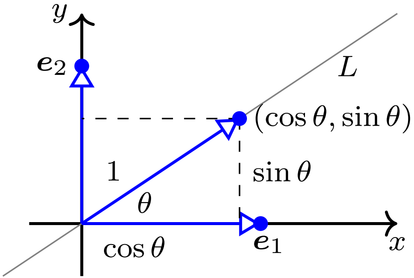
- $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^n$

$$\boldsymbol{b} = \operatorname{proj}_{\boldsymbol{a}} \boldsymbol{b} + \boldsymbol{w}_2$$

- $\operatorname{proj}_{\boldsymbol{a}} \boldsymbol{b}$ is called the *vector component of* $\boldsymbol{b}$ *along* $\boldsymbol{a}$
- $\boldsymbol{b} - \operatorname{proj}_{\boldsymbol{a}} \boldsymbol{b}$ is called the *vector component of* $\boldsymbol{b}$ *orthogonal to* $\boldsymbol{a}$

Orthogonal projection on a line

- Find the orthogonal projections of the vectors $\boldsymbol{e}_1 = \begin{pmatrix}1, & 0\end{pmatrix}$ and $\boldsymbol{e}_2 = \begin{pmatrix}0, & 1\end{pmatrix}$ on the line L that makes an angle θ with the positive $x-$axis in $\mathbb{R}^2$



- First we find the orthogonal projection of $\boldsymbol{e}_1$ onto $\boldsymbol{a} := \begin{pmatrix}\cos\theta, & \sin\theta\end{pmatrix}$

$$\begin{aligned}\mathrm{proj}_{\boldsymbol{a}}\,\boldsymbol{e}_1 &= \frac{\boldsymbol{e}_1\cdot\boldsymbol{a}}{\|\boldsymbol{a}\|^2}\boldsymbol{a} = \frac{\cos\theta+0}{1}\begin{pmatrix}\cos\theta, & \sin\theta\end{pmatrix}\\ &= \begin{pmatrix}\cos^2\theta, & \sin\theta\cos\theta\end{pmatrix}\end{aligned}$$

- We note that for any other vector, $\boldsymbol{u}$ on the line L, $\boldsymbol{u} = \alpha\boldsymbol{a}$ for some $\alpha \in \mathbb{R}$

$$\mathrm{proj}_{\boldsymbol{u}}\,\boldsymbol{e}_1 = \frac{\alpha\boldsymbol{e}_1\cdot\boldsymbol{a}}{\alpha^2\|\boldsymbol{a}\|^2}(\alpha\boldsymbol{a}) = \frac{\boldsymbol{e}_1\cdot\boldsymbol{a}}{\|\boldsymbol{a}\|^2}\boldsymbol{a}$$

- Similarly

$$\begin{aligned}\mathrm{proj}_{\boldsymbol{u}}\,\boldsymbol{e}_2 &= \frac{\boldsymbol{e}_1\cdot\boldsymbol{a}}{\|\boldsymbol{a}\|^2}\boldsymbol{a}\\ &= \begin{pmatrix}\sin\theta\cos\theta, & \sin^2\theta\end{pmatrix}\end{aligned}$$

Matrices

$$A = [1] \in \mathcal{M}_{1\times 1}, \quad B = \begin{pmatrix} 1, & 2 \end{pmatrix} \in \mathcal{M}_{1\times 2}, \quad C = \begin{pmatrix} 3 \\ 5 \end{pmatrix} \in \mathcal{M}_{2\times 1}, \quad D = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \in \mathcal{M}_{2\times 2}$$

$$E = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \in \mathcal{M}_{2\times 3}$$

Special matrices

upper triangular matrix $A = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{pmatrix}$

lower triangular matrix $B = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 0 & -3 \end{pmatrix}$

diagonal matrix $C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \quad D = \begin{pmatrix} d_{11} & 0 & \dots & 0 \\ 0 & d_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_{nn} \end{pmatrix}$

zero matrix $O = \begin{pmatrix} 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}$

Transpose of a matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}, \quad A^\top = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$$

Matrix addition

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 0 & 2 \\ 3 & 5 & 1 \\ -2 & 2 & -3 \end{pmatrix}$$

$$A + B = B + A = \begin{pmatrix} 1 + (-1) & 2 + 0 & 3 + 2 \\ 4 + 3 & 5 + 5 & 6 + 1 \\ 7 + (-2) & 8 + 2 & 9 + (-3) \end{pmatrix} = \begin{pmatrix} 0 & 2 & 5 \\ 7 & 10 & 7 \\ 5 & 10 & 6 \end{pmatrix}$$

Matrix subtraction

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 0 & 2 \\ 3 & 5 & 1 \\ -2 & 2 & -3 \end{pmatrix}$$

$$A - B = -(B - A) = \begin{pmatrix} 1-(-1) & 2-0 & 3-2 \\ 4-3 & 5-5 & 6-1 \\ 7-(-2) & 8-2 & 9-(-3) \end{pmatrix} = \begin{pmatrix} 2 & 2 & 1 \\ 1 & 0 & 5 \\ 9 & 6 & 12 \end{pmatrix}$$

Matrix multiplication

$A \in \mathcal{M}_{m\times n}$, $B \in \mathcal{M}_{n\times r}$, $C = AB \in \mathcal{M}_{m\times r}$

$$A = \begin{pmatrix} 1 & 1 \\ 2 & 2 \\ 3 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & -2 \\ -4 & 5 & 6 \end{pmatrix}$$

$$AB = \begin{pmatrix} 1\times 1 + 1\times(-4) & 1\times 0 + 1\times 1.5 & 1\times(-2) + 1\times 6 \\ 2\times 1 + 2\times(-4) & 2\times 0 + 2\times 1.5 & 2\times(-2) + 2\times 6 \\ 3\times 1 + 3\times(-4) & 3\times 0 + 3\times 1.5 & 3\times(-2) + 3\times 6 \end{pmatrix} = \begin{pmatrix} -3 & 5 & 4 \\ -6 & 10 & 8 \\ -9 & 15 & 12 \end{pmatrix}$$

$$BA = \begin{pmatrix} 1\times 1 + 0\times 2 + 2\times 3 & 1\times 1 + 0\times 2 + (-2)\times 3 \\ -4\times 1 + 5\times 2 + 6\times 3 & -4\times 1 + 5\times 2 + 6\times 3 \end{pmatrix} = \begin{pmatrix} -5 & -5 \\ 24 & 24 \end{pmatrix}$$

Matrix multiplication

$$C = \begin{pmatrix} 1 & -1 \\ 5 & 3 \end{pmatrix}$$

$$C^2 = \begin{pmatrix} 1 & -1 \\ 5 & 3 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 5 & 3 \end{pmatrix} = \begin{pmatrix} 1 \times 1 + (-1) \times 5 & 1 \times (-1) + (-1) \times 3 \\ 5 \times 1 + 3 \times 5 & 5 \times (-1) + 3 \times 3 \end{pmatrix} = \begin{pmatrix} -4 & -4 \\ 20 & 4 \end{pmatrix}$$

$$CI_2 = \begin{pmatrix} 1 & -1 \\ 5 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 \times 1 + (-1) \times 0 & 1 \times 0 + (-1) \times 1 \\ 5 \times 1 + 3 \times 0 & 5 \times 0 + 3 \times 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 5 & 3 \end{pmatrix}$$

$$I_2C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 5 & 3 \end{pmatrix} = \begin{pmatrix} 1 \times 1 + 0 \times 5 & 1 \times (-1) + 0 \times 3 \\ 0 \times 1 + 1 \times 5 & 0 \times (-1) + 1 \times 3 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 5 & 3 \end{pmatrix}$$

Matrix multiplication

$$A = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & 0 & 0 \\ -3 & 1 & 0 \\ 4 & 5 & 2 \end{pmatrix}, \quad AB \neq BA$$

$$\begin{aligned} AB &= \begin{pmatrix} 1\times 3 + 1\times(-3) + (-1)\times 4 & 1\times 0 + 1\times 1 + (-1)\times 5 & 1\times 0 + 1\times 0 + (-1)\times 2 \\ 0\times 3 + 1\times(-3) + 2\times 4 & 0\times 0 + 1\times 1 + 2\times 5 & 0\times 0 + 1\times 0 + 2\times 2 \\ 0\times 3 + 0\times(-3) + 1\times 4 & 0\times 0 + 0\times 1 + 1\times 5 & 0\times 0 + 0\times 0 + 1\times 2 \end{pmatrix} \\ &= \begin{pmatrix} -4 & -4 & -2 \\ 5 & 11 & 4 \\ 4 & 5 & 2 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} BA &= \begin{pmatrix} 3\times 1 + 0\times 0 + 0\times 0 & 3\times 1 + 0\times 1 + 0\times 0 & 3\times(-1) + 0\times 2 + 0\times 1 \\ -3\times 1 + 1\times 0 + 0\times 0 & -3\times 1 + 1\times 1 + 0\times 0 & -3\times(-1) + 1\times 2 + 0\times 1 \\ 4\times 1 + 5\times 0 + 2\times 0 & 4\times 1 + 5\times 1 + 2\times 0 & 4\times(-1) + 5\times 2 + 2\times 1 \end{pmatrix} \\ &= \begin{pmatrix} 3 & 3 & -3 \\ -3 & -2 & 5 \\ 4 & 8 & 8 \end{pmatrix} \end{aligned}$$

Properties of triangular matrices

Theorem

1. *The transpose of a lower triangular matrix is upper triangular, and the transpose of an upper triangular matrix is lower triangular.*
2. *The product of lower triangular matrices is lower triangular, and the product of upper triangular matrices is upper triangular.*

Proof.

- 1 is trivial. We will prove 2.
- Let $A = (a_{ij})$, $B = (b_{ij}) \in \mathcal{M}_{n\times n}$ be lower triangular. Suppose $C = (c_{ij}) = AB$.
- When $i < j$

$$c_{ij} = (a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{i(j-1)}b_{(j-1)j}) + (a_{ij}b_{jj} + \cdots + a_{in}b_{nj})$$

- In the first grouping of terms, b factors are zero since B is lower triangular
- In the second grouping of terms, a factors are zero since A is lower triangular