Tutorial 7

Fundamental spaces and decompositions

Question 1. Determine whether $\boldsymbol{b}$ is in the column space of A, and if so, express $\boldsymbol{b}$ as a linear combination of the column vectors of A

1. $A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{pmatrix}, \quad \boldsymbol{b} = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$

2. $A = \begin{pmatrix} 1 & -1 & 1 \\ 9 & 3 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad \boldsymbol{b} = \begin{pmatrix} 5 \\ 1 \\ -1 \end{pmatrix}$

3. $A = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}, \quad \boldsymbol{b} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$

4. $A = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 2 & 1 \\ 1 & 2 & 1 & 3 \\ 0 & 1 & 2 & 2 \end{pmatrix}, \quad \boldsymbol{b} = \begin{pmatrix} 4 \\ 3 \\ 5 \\ 7 \end{pmatrix}$

Question 2. Suppose that $x_1 = 3, x_2 = 0, x_3 = -1, x_4 = 5$ is a solution of a nonhomogeneous linear system $A\boldsymbol{x} = \boldsymbol{b}$ and that the solution set of the homogeneous system $A\boldsymbol{x} = \boldsymbol{0}$ is given by the formulas

$$x_1 = 5r - 2s, \quad x_2 = s, \quad x_3 = s + t, \quad x_4 = t$$

1. Find a vector form of the general solution of $A\boldsymbol{x} = \boldsymbol{0}$.
2. Find a vector form of the general solution of $A\boldsymbol{x} = \boldsymbol{b}$.

Question 3. Suppose that $x_1 = -1, x_2 = 2, x_3 = 4, x_4 = -3$ is a solution of a nonhomogeneous linear system $A\boldsymbol{x} = \boldsymbol{b}$ and that the solution set of the homogeneous system $A\boldsymbol{x} = \boldsymbol{0}$ is given by the formulas

$$x_1 = -3r + 4s, \quad x_2 = r - s, \quad x_3 = r, \quad x_4 = s$$

1. Find a vector form of the general solution of $A\boldsymbol{x} = \boldsymbol{0}$.
2. Find a vector form of the general solution of $A\boldsymbol{x} = \boldsymbol{b}$.

Question 4. Find the vector form of the general solution of the linear system $A\boldsymbol{x} = \boldsymbol{b}$, and then use that result to find the vector form of the general solution of $A\boldsymbol{x} = \boldsymbol{0}$.

1.
$$\begin{aligned} x_1 - 3x_3 &= 1 \\ 2x_1 - 6x_2 &= 2 \end{aligned}$$

2.
$$\begin{aligned} x_1 + x_2 + 2x_3 &= 5 \\ x_1 + x_3 &= -2 \\ 2x_1 + x_2 + 3x_3 &= 3 \end{aligned}$$

1. The reduced row echelon form of the augmented matrix is:

$$\left(\begin{array}{ccc|c} 1 & 0 & -3 & 1 \\ 0 & 1 & -1 & 0 \end{array}\right)$$

The general solution of the system is

$$x_1 = 3t + 1, \quad x_2 = t, \quad x_3 = t.$$

A vector form of the general solution of $A\boldsymbol{x} = \boldsymbol{b}$ is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}.$$

A vector form of the general solution of $A\boldsymbol{x} = \boldsymbol{0}$ is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = t \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}.$$

2. The reduced row echelon form of the augmented matrix is:

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & -2 \\ 0 & 1 & 1 & 7 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

The general solution of the system is

$$x_1 = -t - 2, \quad x_2 = 7 - t, \quad x_3 = t.$$

A vector form of the general solution of $A\boldsymbol{x} = \boldsymbol{b}$ is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2 \\ 7 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}.$$

A vector form of the general solution of $A\boldsymbol{x} = \boldsymbol{0}$ is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = t \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}.$$

Question 5. Find bases for the null space and row space of A.

1. $A = \begin{pmatrix} 1 & -1 & 3 \\ 5 & -4 & -4 \\ 7 & -6 & 2 \end{pmatrix}$

2. $A = \begin{pmatrix} 2 & 0 & -1 \\ 4 & 0 & -2 \\ 0 & 0 & 0 \end{pmatrix}$

3. $A = \begin{pmatrix} 1 & 4 & 5 & 2 \\ 2 & 1 & 3 & 0 \\ -1 & 3 & 2 & 2 \end{pmatrix}$

4. $A = \begin{pmatrix} 1 & 4 & 5 & 6 & 9 \\ 3 & -2 & 1 & 4 & -1 \\ -1 & 0 & -1 & -2 & -1 \\ 2 & 3 & 5 & 7 & 8 \end{pmatrix}$

Question 6. By inspection, find a basis for the row space and for the column space of the given matrix

1. $A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

2. $A = \begin{pmatrix} 1 & -3 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

3. $A = \begin{pmatrix} 1 & 2 & 4 & 5 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

4. $A = \begin{pmatrix} 1 & 2 & -1 & 5 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 1 & -7 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Question 7. Construct a matrix whose null space consists of all linear combinations of the vectors

$$\boldsymbol{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 3 \\ 2 \end{pmatrix}, \quad \text{and} \quad \boldsymbol{v}_2 = \begin{pmatrix} 2 \\ 0 \\ -2 \\ 4 \end{pmatrix}$$

Question 8. Find the rank and nullity of the matrix A by reducing it to row echelon form.

1. $A = \begin{pmatrix} 1 & 2 & -1 & 1 \\ 2 & 4 & -2 & 2 \\ 3 & 6 & -3 & 3 \\ 4 & 8 & -4 & 4 \end{pmatrix}$

2. $A = \begin{pmatrix} 1 & -2 & 2 & 3 & -1 \\ -3 & 6 & -1 & 1 & -7 \\ 2 & -4 & 5 & 8 & -4 \end{pmatrix}$

3. $A = \begin{pmatrix} 1 & 0 & -2 & 1 & 0 \\ 0 & -1 & -3 & 1 & 3 \\ -2 & -1 & 1 & -1 & 3 \\ 0 & 1 & 3 & 0 & -4 \end{pmatrix}$

4. $A = \begin{pmatrix} 1 & 3 & 1 & 3 \\ 0 & 1 & 1 & 0 \\ -3 & 0 & 6 & -1 \\ 3 & 4 & -2 & 1 \\ 2 & 0 & -4 & -2 \end{pmatrix}$

1. The reduced row echelon form of A is

$$\begin{pmatrix} 1 & 2 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

rank $(A) = 1$, nullity $(A) = 3$.

2. The reduced row echelon form of A is

$$\begin{pmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

rank $(A) = 2$, nullity $(A) = 3$.

3. The reduced row echelon form of A is

$$\begin{pmatrix} 1 & 0 & -2 & 0 & 1 \\ 0 & 1 & 3 & 0 & -4 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$\operatorname{rank}(A) = 3$, $\operatorname{nullity}(A) = 2$.

4. The reduced row echelon form of A is

$$\begin{pmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\operatorname{rank}(A) = 3$, $\operatorname{nullity}(A) = 1$.

Question 9. The matrix R is the reduced row echelon form of the matrix A.

(a) By inspection of the matrix R, find the rank and nullity of A.

(b) Find the number of leading variables and the number of parameters in the general solution of $A\boldsymbol{x} = \mathbf{0}$ without solving the system.

1. $A = \begin{pmatrix} 2 & -1 & -3 \\ -1 & 2 & -3 \\ 1 & 1 & 4 \end{pmatrix}$ $\quad R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

2. $A = \begin{pmatrix} 2 & -1 & -3 \\ -1 & 2 & -3 \\ 1 & 1 & -6 \end{pmatrix}$ $\quad R = \begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{pmatrix}$

3. $A = \begin{pmatrix} 2 & -1 & -3 \\ -2 & 1 & 3 \\ -4 & 2 & 6 \end{pmatrix}$ $\quad R = \begin{pmatrix} 1 & -\frac{1}{2} & -\frac{3}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

4. $A = \begin{pmatrix} 0 & 2 & 2 & 4 \\ 1 & 0 & -1 & -3 \\ 2 & 3 & 1 & 1 \\ -2 & 1 & 3 & -2 \end{pmatrix}$ $\quad R = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

Question 10. Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be the linear transformation defined by the formula

$$T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 + 3x_2 \\ x_1 - x_2 \\ x_1 \end{pmatrix}.$$

1. Find the rank of the standard matrix for T.

2. Find the nullity of the standard matrix for T.

Question 11. Let $T : \mathbb{R}^5 \to \mathbb{R}^3$ be the linear transformation defined by the formula

$$T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ x_2 + x_3 + x_4 \\ x_4 + x_5 \end{pmatrix}$$

1. Find the rank of the standard matrix for T.
2. Find the nullity of the standard matrix for T.

Question 12. Discuss how the rank of A varies with t.

1. $A = \begin{pmatrix} 1 & 1 & t \\ 1 & 1 & 1 \\ t & 1 & 1 \end{pmatrix}$
2. $A = \begin{pmatrix} t & 3 & -1 \\ 3 & 6 & -2 \\ -1 & -3 & t \end{pmatrix}$

Question 13. Are there values of r and s for which

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r-2 & 2 \\ 0 & s-1 & r+2 \\ 0 & 0 & 3 \end{pmatrix}$$

has rank 1? Has rank 2? If so, find those values.

Question 14.

1. If A is a 3×5 matrix, then the rank of A is at most _____.
2. If A is a 3×5 matrix, then the nullity of A is at most _____.
3. If A is a 3×5 matrix, then the rank of $A^\top$ is at most _____.
4. If A is a 5×3 matrix, then the nullity of $A^\top$ is at most _____.

Question 15.

1. If A is a 3×5 matrix, then the number of leading 1's in the reduced row echelon form of A is at most _____.

2. If A is a 3×5 matrix, then the number of parameters in the general solution of $A\boldsymbol{x} = \mathbf{0}$ is at most _____.

3. If A is a 5×3 matrix, then the number of leading 1's in the reduced row echelon form of A is at most _____.

4. If A is a 5×3 matrix, then the number of parameters in the general solution of $A\boldsymbol{x} = \mathbf{0}$ is at most _____.

Question 16. Let

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}.$$

Show that A has rank 2 if and only if one or more of the following determinants is nonzero:

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, \quad \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}, \quad \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}.$$

Question 17. Determine whether the matrix operator $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by the equations is bijective.

1.
$$\begin{aligned} w_1 &= x_1 + 2x_2 \\ w_2 &= -x_1 + x_2 \end{aligned}$$

2.
$$\begin{aligned} w_1 &= 4x_1 - 6x_2 \\ w_2 &= -2x_1 + 3x_2 \end{aligned}$$

3.
$$\begin{aligned} w_1 &= x_1 - 2x_2 + 2x_3 \\ w_2 &= 2x_1 + x_2 + x_3 \\ w_3 &= x_1 + x_2 \end{aligned}$$

4.
$$\begin{aligned} w_1 &= x_1 - 3x_2 + 4x_3 \\ w_2 &= -x_1 + x_2 + x_3 \\ w_3 &= -2x_2 + 5x_3 \end{aligned}$$

Question 18. Determine whether multiplication by A is an injective matrix transformation.

1. $A = \begin{pmatrix} 1 & -1 \\ 2 & 0 \\ 3 & -4 \end{pmatrix}$

2. $A = \begin{pmatrix} 1 & 2 & 3 \\ -1 & 0 & -4 \end{pmatrix}$

3. $A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix}$

4. $A = \begin{pmatrix} 4 & 3 \\ 1 & 1 \end{pmatrix}$

Question 19. Let T_A be multiplication by the matrix A. Find:

(a) a basis for the range of T_A.

(b) a basis for the kernel of T_A.

(d) the rank and nullity of A.

1. $A = \begin{pmatrix} 1 & -1 & 3 \\ 5 & 6 & -4 \\ 7 & 4 & 2 \end{pmatrix}$

2. $A = \begin{pmatrix} 2 & 0 & -1 \\ 4 & 0 & -2 \\ 20 & 0 & 0 \end{pmatrix}$

Question 20. Let $T_A : \mathbb{R}^4 \to \mathbb{R}^3$ be multiplication by A. Find:

(a) a basis for the kernel of T_A.

(b) a basis for the range of T_A that consists of column vectors of A.

1. $A = \begin{pmatrix} 1 & 2 & -1 & -2 \\ -3 & 1 & 3 & 4 \\ -3 & 8 & 4 & 2 \end{pmatrix}$

2. $A = \begin{pmatrix} 1 & 1 & 0 & 1 \\ -2 & 4 & 2 & 2 \\ -1 & 8 & 3 & 5 \end{pmatrix}$

Question 21. Let A be an $n \times n$ matrix such that $\det(A) = 0$, and let $T : \mathbb{R}^n \to \mathbb{R}^n$ be multiplication by A.

1. What can you say about the range of the matrix operator T? Give an example that illustrates your conclusion.

2. What can you say about the number of vectors that T maps into $\mathbf{0}$?

What if $\det(A) \neq 0$?

Question 22. Confirm by multiplication that $\boldsymbol{x}$ is an eigenvector of A, and find the corresponding eigenvalue.

1. $A = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}, \quad \boldsymbol{x} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

2. $A = \begin{pmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 0 & 4 \end{pmatrix}, \quad \boldsymbol{x} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$

3. $A = \begin{pmatrix} 5 & -1 \\ 1 & 3 \end{pmatrix}, \quad \boldsymbol{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

4. $A = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}, \quad \boldsymbol{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

Question 23. Find the characteristic equation, the eigenvalues, and the bases for the eigenspaces of the matrix A.

1. $A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$
2. $A = \begin{pmatrix} -2 & -7 \\ 1 & 2 \end{pmatrix}$
3. $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
4. $A = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$
5. $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$
6. $A = \begin{pmatrix} 2 & -3 \\ 0 & 2 \end{pmatrix}$
7. $A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$
8. $A = \begin{pmatrix} 1 & 2 \\ -2 & -1 \end{pmatrix}$
9. $A = \begin{pmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$
10. $A = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & 4 \end{pmatrix}$
11. $A = \begin{pmatrix} 6 & 3 & -8 \\ 0 & -2 & 0 \\ 1 & 0 & -3 \end{pmatrix}$
12. $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$
13. $A = \begin{pmatrix} 4 & 0 & -1 \\ 0 & 3 & 0 \\ 1 & 0 & 2 \end{pmatrix}$
14. $A = \begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix}$

Question 24. Find the characteristic equation of the matrix by inspection.

1. $A = \begin{pmatrix} 3 & 0 & 0 \\ -2 & 7 & 0 \\ 4 & 8 & 1 \end{pmatrix}$
2. $A = \begin{pmatrix} 9 & -8 & 6 & 3 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 7 \end{pmatrix}$

Question 25. Find the eigenvalues and the corresponding eigenspaces of the stated matrix operator on $\mathbb{R}^2$. Use geometric reasoning to help finding the answers.

1. Reflection about the line $y = x$.
2. Orthogonal projection onto the x-axis.
3. Rotation about the origin through a positive angle of $90°$.

4. Contraction with factor α $(0 < \alpha < 1)$.
5. Shear in the $x-$direction by a factor α $(\alpha \neq 0)$.
6. Reflection about the $y-$axis.
7. Rotation about the origin through a positive angle of 180°.
8. Dilation with factor α $(\alpha > 1)$.
9. Expansion in the $y-$direction with factor α $(\alpha > 1)$.
10. Shear in the $y-$direction by a factor α $(\alpha \neq 0)$.

Question 26. Find the eigenvalues and the corresponding eigenspaces of the stated matrix operator on $\mathbb{R}^3$. Use geometric reasoning to find the answers.

1. Reflection about the xy-plane.
2. Orthogonal projection onto the xz-plane.
3. Counterclockwise rotation about the positive x-axis through an angle of 90°.
4. Contraction with factor α $(0 \leq \alpha < 1)$.
5. Reflection about the xz-plane.
6. Orthogonal projection onto the yz-plane.
7. Counterclockwise rotation about the positive y-axis through an angle of 180°.
8. Dilation with factor α $(\alpha > 1)$.

Question 27. Find $\det(A)$ given that A has $p(\lambda)$ as its characteristic polynomial.

1. $p(\lambda) = \lambda^3 - 2\lambda^2 + \lambda + 5$
2. $p(\lambda) = \lambda^4 - \lambda^3 + 7$

Question 28. Suppose that the characteristic polynomial of some matrix A is found to be

$$p(\lambda) = (\lambda - 1)(\lambda - 3)^2(\lambda - 4)^3.$$

1. What is the size of A?
2. Is A invertible?
3. How many eigenspaces does A have?

Question 29. The eigenvectors that we have been studying are sometimes called *right eigenvectors* to distinguish them from *left eigenvectors*, which are $n \times 1$ column vectors $\boldsymbol{x}$ that satisfy the equation

$$\boldsymbol{x}^\top A = \mu \boldsymbol{x}^\top$$

for some scalar μ. For a given matrix A, how are the right eigenvectors and their corresponding eigenvalues related to the left eigenvectors and their corresponding eigenvalues?

Question 30. Prove that the characteristic equation of a 2×2 matrix A can be expressed as

$$\lambda^2 - \operatorname{tr}(A)\lambda + \det(A) = 0,$$

where $\operatorname{tr}(A)$ is the trace of A.

Question 31. Use the result from the Question 30 to show that if

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

then the solutions of the characteristic equation of A are

$$\lambda = \frac{1}{2}\left((a+d) \pm \sqrt{(a-d)^2 + 4bc}\right).$$

Use this result to show that A has

(a) two distinct real eigenvalues if $(a-d)^2 + 4bc > 0$.

(b) two repeated real eigenvalues if $(a-d)^2 + 4bc = 0$.

(c) complex conjugate eigenvalues if $(a-d)^2 + 4bc < 0$.

Question 32. Let A be the matrix in from the Question 31. Show that if $b \neq 0$, then

$$\boldsymbol{x}_1 = \begin{pmatrix} -b \\ a - \lambda_1 \end{pmatrix}, \quad \boldsymbol{x}_2 = \begin{pmatrix} -b \\ a - \lambda_2 \end{pmatrix}$$

are eigenvectors of A that correspond, respectively, to the eigenvalues

$$\lambda_1 = \frac{1}{2}\left[(a+d) + \sqrt{(a-d)^2 + 4bc}\right]$$

and

$$\lambda_2 = \frac{1}{2}\left[(a+d) - \sqrt{(a-d)^2 + 4bc}\right].$$

Question 33. Use the result of Question 30 to prove that if

$$p(\lambda) = \lambda^2 + c_1\lambda + c_2$$

is the characteristic polynomial of a 2×2 matrix, then

$$p(A) = A^2 + c_1 A + c_2 I = O.$$

(Stated informally, A satisfies its characteristic equation. This result is true as well for $n \times n$ matrices.)

Question 34. Prove: If a, b, c, d are integers such that $a + b = c + d$, then

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

has integer eigenvalues.

Question 35. Prove: If λ is an eigenvalue of an invertible matrix A and $\boldsymbol{x}$ is a corresponding eigenvector, then $1/\lambda$ is an eigenvalue of A^{-1} and $\boldsymbol{x}$ is a corresponding eigenvector.

Question 36. Prove: If λ is an eigenvalue of A, $\boldsymbol{x}$ is a corresponding eigenvector, and s is a scalar, then $\lambda - s$ is an eigenvalue of $A - sI$ and $\boldsymbol{x}$ is a corresponding eigenvector.

Question 37. Prove: If λ is an eigenvalue of A and $\boldsymbol{x}$ is a corresponding eigenvector, then $s\lambda$ is an eigenvalue of sA for every scalar s and $\boldsymbol{x}$ is a corresponding eigenvector.

Question 38. Find the eigenvalues and bases for the eigenspaces of

$$A = \begin{pmatrix} -2 & 2 & 3 \\ -2 & 3 & 2 \\ -4 & 2 & 5 \end{pmatrix}$$

and then use Questions 35 and 36 to find the eigenvalues and bases for the eigenspaces of

1. A^{-1}
2. $A - 3I$
3. $A + 2I$

Question 39. Show that A and B are not similar matrices.

1. $A = \begin{pmatrix} 1 & 1 \\ 3 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ 3 & -2 \end{pmatrix}$
2. $A = \begin{pmatrix} 4 & -1 \\ 2 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 4 & 1 \\ 2 & 4 \end{pmatrix}$
3. $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 2 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
4. $A = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 0 & 2 \\ 3 & 0 & 3 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 2 & 0 \\ 0 & 1 & 1 \end{pmatrix}$

Question 40.

A Procedure for Diagonalizing an $n \times n$ Matrix
Step 1. Determine first whether the matrix is actually diagonalizable by searching for n linearly independent eigenvectors. One way to do this is to find a basis for each eigenspace and count the total number of vectors obtained. If there is a total of n vectors, then the matrix is diagonalizable, and if the total is less than n, then it is not.
Step 2. If you ascertained that the matrix is diagonalizable, then form the matrix

$$P = [\boldsymbol{v}_1 \quad \boldsymbol{v}_2 \quad \cdots \quad \boldsymbol{v}_n]$$

whose column vectors are the n basis vectors you obtained in Step 1.
Step 3. $P^{-1}AP$ will be a diagonal matrix whose successive diagonal entries are the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ that correspond to the successive columns of P.

Following the above procedure to find a matrix P that diagonalizes A, and check your work by computing $P^{-1}AP$.

1. $A = \begin{pmatrix} 1 & 0 \\ 6 & -1 \end{pmatrix}$
2. $A = \begin{pmatrix} -14 & 12 \\ -20 & 17 \end{pmatrix}$
3. $A = \begin{pmatrix} 2 & 0 & -2 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$
4. $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$

Question 41. Let

1. $A = \begin{pmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 0 & 4 \end{pmatrix}$
2. $A = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 2 \end{pmatrix}$

(a) Find the eigenvalues of A.

(b) For each eigenvalue λ, find the rank of the matrix $\lambda I - A$.

(c) Is A diagonalizable? Justify your conclusion.

For Questions 42 – 44 Find an LU-decomposition of the coefficient matrix - try both Doolittle decomposition and Crout decomposition. Then solve the system using the decomposition.

Question 42.

1. $\begin{pmatrix} 2 & 8 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -2 \\ -2 \end{pmatrix}$

2. $\begin{pmatrix} -5 & -10 \\ 6 & 5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -10 \\ 19 \end{pmatrix}$

3. $\begin{pmatrix} 2 & -2 & -2 \\ 0 & -2 & 2 \\ -1 & 5 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -4 \\ -2 \\ 6 \end{pmatrix}$

4. $\begin{pmatrix} -3 & 12 & -6 \\ 1 & -2 & 2 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -33 \\ 7 \\ -1 \end{pmatrix}$

5. $\begin{pmatrix} 2 & 6 & 2 \\ -3 & -8 & 0 \\ 4 & 9 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix}$

6. $\begin{pmatrix} 3 & -6 & -3 \\ 2 & 0 & 6 \\ -4 & 7 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ -22 \\ 3 \end{pmatrix}$

7. $\begin{pmatrix} -1 & -3 & 2 \\ -6 & -19 & 10 \\ 3 & 9 & -5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -9 \\ -59 \\ 28 \end{pmatrix}$

8. $\begin{pmatrix} -4 & 0 & 1 \\ 8 & 2 & -1 \\ -8 & -6 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -5 \\ 15 \\ -46 \end{pmatrix}$

Question 43.

1. $\begin{pmatrix} -2 & -1 & -1 & -2 \\ 2 & 4 & 3 & 1 \\ 4 & 2 & -2 & -1 \\ -1 & 0 & 4 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 3 \\ 11 \\ -10 \\ 8 \end{pmatrix}$

2. $\begin{pmatrix} 2 & -2 & 0 & 3 \\ -2 & 3 & 2 & -3 \\ 4 & -5 & -4 & 10 \\ -4 & 7 & 10 & -12 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 11 \\ -9 \\ 10 \\ 2 \end{pmatrix}$

3. $\begin{pmatrix} 4 & 3 & 4 & -2 \\ 0 & -1 & 3 & 3 \\ 1 & 2 & -3 & 1 \\ 0 & -4 & -2 & -4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -20 \\ 1 \\ 5 \\ 14 \end{pmatrix}$

4. $\begin{pmatrix} -5 & 2 & -2 & 2 \\ -10 & 0 & -8 & 7 \\ 20 & -4 & 10 & -7 \\ -20 & 12 & 0 & -4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 16 \\ 25 \\ -49 \\ 52 \end{pmatrix}$

5. $\begin{pmatrix} -1 & 1 & 0 & -3 \\ 0 & 4 & -4 & -4 \\ 0 & -2 & 4 & 0 \\ 4 & -3 & -1 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 3 \\ -4 \\ -8 \\ -13 \end{pmatrix}$

6. $\begin{pmatrix} 1 & -2 & -3 & -3 \\ 3 & -10 & -12 & -5 \\ 5 & -22 & -26 & 0 \\ 2 & 12 & 0 & -18 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -19 \\ -89 \\ -199 \\ 66 \end{pmatrix}$

7. $\begin{pmatrix} 1 & 0 & -1 & -4 \\ -2 & 1 & -2 & -2 \\ 4 & -3 & 4 & -2 \\ -2 & -4 & -3 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 11 \\ 4 \\ 2 \\ 24 \end{pmatrix}$

8. $\begin{pmatrix} 1 & 1 & -5 & -1 \\ 2 & -1 & -5 & 3 \\ -4 & -16 & 38 & 23 \\ -5 & -20 & 52 & 35 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -19 \\ -25 \\ 122 \\ 158 \end{pmatrix}$

Question 44.

1.

$$\begin{aligned} -x_1 + 5x_2 - 7x_3 &= 14 \\ -3x_1 + 8x_2 - 4x_3 &= 28 \\ x_1 - 3x_2 + 2x_3 &= -10 \end{aligned}$$

2.

$$\begin{aligned} -12x_1 - 24x_2 - 16x_3 &= 8 \\ -3x_1 - 5x_2 - 2x_3 &= 4 \\ 6x_1 + 9x_2 + x_3 &= -11 \end{aligned}$$

3.

$$\begin{aligned} -4x_1 + 14x_2 - 7x_3 &= 107 \\ -x_1 + 2x_2 - x_3 &= 17 \\ 3x_1 - 9x_2 + 7x_3 &= -83 \end{aligned}$$

4.

$$\begin{aligned} 20x_1 - 11x_2 - 29x_3 &= -92 \\ -4x_1 + 2x_2 + 5x_3 &= 15 \\ -8x_1 + 4x_2 + 14x_3 &= 50 \end{aligned}$$

5.

$$\begin{aligned} 4x_1 - 2x_2 &= 4 \\ 2x_1 + 4x_2 &= 2 \\ x_2 - 5x_3 &= 2 \end{aligned}$$

6.

$$\begin{aligned} -9x_1 + 12x_2 - 9x_3 &= 45 \\ -9x_1 + 3x_2 - 6x_3 &= 24 \\ -3x_1 + 3x_2 - 3x_3 &= 12 \end{aligned}$$

7.

$$\begin{aligned} 3x_1 + 27x_2 - 29x_3 &= 28 \\ -x_1 - 4x_2 + 2x_3 + 5x_4 &= -5 \\ -3x_2 - 3x_3 + 13x_4 &= -1 \\ -5x_1 - 23x_2 + 15x_3 + 21x_4 &= -28 \end{aligned}$$

8.

$$\begin{aligned} 4x_1 - 12x_2 - 29x_3 - 12x_4 &= -149 \\ 3x_2 - 4x_3 + x_4 &= -11 \\ x_1 - 3x_2 - x_3 + x_4 &= -18 \\ 3x_1 + 6x_2 - 28x_3 + 4x_4 &= -122 \end{aligned}$$

9.

$$\begin{aligned} -16x_1 + 15x_2 - 21x_3 - 26x_4 &= -86 \\ 20x_2 - 20x_3 - 10x_4 &= -100 \\ 4x_1 + 2x_3 + 4x_4 &= 2 \\ 16x_1 + 5x_2 + 3x_3 + 14x_4 &= -16 \end{aligned}$$

Question 45. Determine A^{-1} using LU decomposition and verify the correctness of the following identity:

$$A = LU \Longrightarrow A^{-1} = (LU)^{-1} = U^{-1}L^{-1}.$$

Try both Doolittle decomposition and Crout decomposition.

1. $\begin{pmatrix} 5 & -2 & -4 \\ 15 & -10 & -7 \\ -10 & -8 & 25 \end{pmatrix}$

2. $\begin{pmatrix} -1 & 2 & 5 \\ -2 & 0 & 12 \\ 1 & -6 & -1 \end{pmatrix}$

3. $\begin{pmatrix} -1 & 5 & -2 \\ -2 & 8 & -8 \\ -1 & 1 & -12 \end{pmatrix}$

4. $\begin{pmatrix} -2 & 1 & 3 \\ 2 & 2 & 2 \\ 0 & -4 & -3 \end{pmatrix}$

5. $\begin{pmatrix} 4 & -4 & 5 \\ 12 & -16 & 14 \\ -16 & 36 & -14 \end{pmatrix}$

6. $\begin{pmatrix} 2 & -5 & 1 \\ -2 & 4 & -2 \\ -2 & 7 & 2 \end{pmatrix}$

7. $\begin{pmatrix} -1 & -4 & 5 \\ 0 & 2 & -4 \\ -5 & -22 & -26 \end{pmatrix}$

8. $\begin{pmatrix} -3 & -5 & -3 \\ -12 & -19 & -10 \\ 0 & 0 & 1 \end{pmatrix}$

9. $\begin{pmatrix} 3 & 3 & 0 & -5 \\ 15 & 16 & 0 & -23 \\ -3 & 2 & 4 & 10 \\ 0 & 2 & -8 & 12 \end{pmatrix}$

10. $\begin{pmatrix} 1 & -5 & 4 & -2 \\ 4 & -17 & 12 & -8 \\ -2 & -2 & 11 & 9 \\ 5 & -37 & 36 & -8 \end{pmatrix}$

11. $\begin{pmatrix} 1 & 1 & -4 & -2 \\ 3 & 2 & -10 & -2 \\ 2 & -1 & -1 & 13 \\ -1 & -4 & 8 & 5 \end{pmatrix}$

12. $\begin{pmatrix} 3 & 2 & -3 & 5 \\ -6 & -9 & 6 & -11 \\ -6 & 6 & 9 & -10 \\ -15 & 10 & 12 & -16 \end{pmatrix}$

Question 46. Determine $\det(A)$ using LU decomposition and verify the correctness of the following identity:

$$A = LU \Longrightarrow \det(A) = \det(L)\det(U).$$

Try both Doolittle decomposition and Crout decomposition.

1. $A = \begin{pmatrix} -1 & 0 & -1 \\ -4 & 1 & -4 \\ -4 & -5 & -4 \end{pmatrix}$

2. $A = \begin{pmatrix} -2 & 4 & 3 \\ -4 & 13 & 6 \\ 8 & -36 & -11 \end{pmatrix}$

3. $A = \begin{pmatrix} 1 & 4 & 1 \\ 3 & 17 & 0 \\ 0 & -20 & 13 \end{pmatrix}$

4. $A = \begin{pmatrix} -3 & -1 & 4 \\ -15 & -4 & 23 \\ 3 & 2 & -6 \end{pmatrix}$

5. $A = \begin{pmatrix} -2 & -2 & 4 \\ 2 & 3 & -9 \\ -2 & -7 & 25 \end{pmatrix}$

6. $A = \begin{pmatrix} -2 & -4 & 0 \\ 0 & -3 & -3 \\ 0 & 9 & 6 \end{pmatrix}$

7. $A = \begin{pmatrix} -2 & 4 & 2 & 0 \\ 6 & -8 & -2 & -1 \\ 8 & -24 & -12 & -3 \\ -8 & 0 & -16 & 13 \end{pmatrix}$

8. $A = \begin{pmatrix} 2 & -2 & 4 & -1 \\ -6 & 11 & -9 & 0 \\ 8 & -33 & 2 & 8 \\ 4 & -4 & 11 & -8 \end{pmatrix}$

9. $A = \begin{pmatrix} 4 & 0 & -4 & 1 \\ 8 & -3 & -3 & 6 \\ -4 & -12 & 23 & 11 \\ -16 & 12 & -2 & -7 \end{pmatrix}$

10. $A = \begin{pmatrix} 3 & -4 & 3 & 5 \\ -15 & 24 & -10 & -30 \\ -12 & 20 & -4 & -20 \\ -6 & 28 & 25 & -23 \end{pmatrix}$