Tutorial 6

Bases and matrix operators

Question 1. Show that the following set of vectors forms a basis for $\mathbb{R}^2$ and $\mathbb{R}^3$ respectively.

1. $\{ (2, \ 1), (3, \ 0) \}$
2. $\{ (3, \ 1, \ -4), (2, \ 5, \ 6) \}, (1, \ 4, \ 8)$

Question 2. Show that the following matrices form a basis for $\mathcal{M}_{\times 2}2$.

1. $\begin{pmatrix} 3 & 6 \\ 3 & -6 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -8 \\ -12 & -4 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix}$
2. $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

Question 3. In each part, show that the set of vectors is not a basis for $\mathbb{R}^3$

1. $\{ (2, \ -3, \ 1), (4, \ 1, \ 1), (0, \ -7, \ 1) \}$
2. $\{ (1, \ 6, \ 4) \}, (2, \ 4, \ -1), (-1, \ 2, \ 5)$

Question 4. Show that the following matrices do not form a basis for $\mathcal{M}_{2\times 2}$.

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 2 & -2 \\ 3 & 2 \end{pmatrix}, \quad \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$$

Question 5. Find the coordinate vector of $\boldsymbol{w}$ relative to the basis $S = \{ \boldsymbol{u}_1, \ \boldsymbol{u}_2 \}$ for $\mathbb{R}^2$.

1. $\boldsymbol{u}_1 = (2, \ -4)$, $\boldsymbol{u}_2 = (3, \ 8)$, $\boldsymbol{w} = (1, \ 1)$
2. $\boldsymbol{u}_1 = (1, \ 1)$, $\boldsymbol{u}_2 = (0, \ 2)$, $\boldsymbol{w} = (a, \ b)$
3. $\boldsymbol{u}_1 = (1, \ -1)$, $\boldsymbol{u}_2 = (1, \ 1)$, $\boldsymbol{w} = (1, \ 0)$
4. $\boldsymbol{u}_1 = (1, \ -1)$, $\boldsymbol{u}_2 = (1, \ 1)$, $\boldsymbol{w} = (0, \ 1)$

Question 6. Find the coordinate vector of $\boldsymbol{u}$ relative to the basis $S = \{ \boldsymbol{v}_1, \ \boldsymbol{v}_2, \ \boldsymbol{v}_3 \}$ for $\mathbb{R}^3$

1. $\boldsymbol{v}_1 = \begin{pmatrix}1, & 0, & 0\end{pmatrix}$, $\boldsymbol{v}_2 = \begin{pmatrix}2, & 2, & 0\end{pmatrix}$, $\boldsymbol{v}_3 = \begin{pmatrix}3, & 3, & 3\end{pmatrix}$, $\boldsymbol{u} = \begin{pmatrix}2, & -1, & 3\end{pmatrix}$
2. $\boldsymbol{v}_1 = \begin{pmatrix}1, & 2, & 3\end{pmatrix}$, $\boldsymbol{v}_2 = \begin{pmatrix}-4, & 5, & 6\end{pmatrix}$, $\boldsymbol{v}_3 = \begin{pmatrix}7, & -8, & 9\end{pmatrix}$, $\boldsymbol{u} = \begin{pmatrix}5, & -12, & 3\end{pmatrix}$

Question 7. For each case, first show that the set $S = \{ A_1, A_2, A_3, A_4 \}$ is a basis for $\mathcal{M}_{2\times 2}$, then express A as a linear combination of the vectors in S, and then find the coordinate vector of A relative to S.

1. $A_1 = \begin{pmatrix}1 & 1\\1 & 1\end{pmatrix}$, $A_2 = \begin{pmatrix}0 & 1\\1 & 1\end{pmatrix}$, $A_3 = \begin{pmatrix}0 & 0\\1 & 1\end{pmatrix}$, $A_4 = \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}$; $A = \begin{pmatrix}1 & 0\\1 & 0\end{pmatrix}$
2. $A_1 = \begin{pmatrix}1 & 0\\1 & 0\end{pmatrix}$, $A_2 = \begin{pmatrix}1 & 1\\0 & 0\end{pmatrix}$, $A_3 = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$, $A_4 = \begin{pmatrix}0 & 0\\1 & 0\end{pmatrix}$; $A = \begin{pmatrix}6 & 2\\5 & 3\end{pmatrix}$

Question 8. In each part, let $T_A : \mathbb{R}^3 \to \mathbb{R}^3$ be multiplication by A, and let $\{ \boldsymbol{e}_1, \ \boldsymbol{e}_2, \ \boldsymbol{e}_3 \}$ be the standard basis for $\mathbb{R}^3$. Determine whether the set $\{ T_A(\boldsymbol{e}_1), \ T_A(\boldsymbol{e}_2), \ T_A(\boldsymbol{e}_3) \}$ is linearly independent in $\mathbb{R}^3$

1. $A = \begin{pmatrix}1 & 1 & 1\\0 & 1 & -3\\-1 & 2 & 0\end{pmatrix}$
2. $A = \begin{pmatrix}1 & 1 & 2\\0 & 1 & 1\\-1 & 2 & 1\end{pmatrix}$

Question 9. In each part, let $T_A : \mathbb{R}^3 \to \mathbb{R}^3$ be multiplication by A, and let $\boldsymbol{u} = \begin{pmatrix}1, & -2, & -1\end{pmatrix}$. Find the coordinate vector of $T_A(\boldsymbol{u})$ relative to the basis $S = \{ \begin{pmatrix}1, & 1, & 0\end{pmatrix}, \ \begin{pmatrix}0, & 1, & 1\end{pmatrix}, \ \begin{pmatrix}1, & 1, & 1\end{pmatrix} \}$ for $\mathbb{R}^3$.

1. $A = \begin{pmatrix}2 & -1 & 0\\1 & 1 & 1\\0 & -1 & 2\end{pmatrix}$
2. $A = \begin{pmatrix}0 & 1 & 0\\1 & 0 & 1\\0 & 0 & 1\end{pmatrix}$

Question 10. Find a basis for the solution space of the homogeneous linear system, and find the dimension of that space

1.

$$\begin{aligned} x_1 + x_2 - x_3 &= 0 \\ -2x_1 - x_2 + 2x_3 &= 0 \\ -x_1 + x_3 &= 0 \end{aligned}$$

2.

$$\begin{aligned} 3x_1 + x_2 + x_3 + x_4 &= 0 \\ 5x_1 - x_2 + x_3 - x_4 &= 0 \end{aligned}$$

3.

$$\begin{aligned} 2x_1 + x_2 + 3x_3 &= 0 \\ x_1 + 5x_3 &= 0 \\ x_2 + x_3 &= 0 \end{aligned}$$

4.

$$\begin{aligned} x_1 - 3x_2 + x_3 &= 0 \\ 2x_1 - 6x_2 + 2x_3 &= 0 \\ 3x_1 - 9x_2 + 3x_3 &= 0 \end{aligned}$$

5.

$$\begin{aligned} x_1 - 4x_2 + 3x_3 - x_4 &= 0 \\ 2x_1 - 8x_2 + 6x_3 - 2x_4 &= 0 \end{aligned}$$

6.

$$\begin{aligned} x + y + z &= 0 \\ 3x + 2y - 2z &= 0 \\ 4x + 3y - z &= 0 \\ 6x + 5y + z &= 0 \end{aligned}$$

Question 11. In each part, find a basis for the given subspace of $\mathbb{R}^3$, and sate its dimension

1. The plane $3x - 2y + 5z = 0$.
2. The plane $x - y = 0$.
3. The line $x = 2t$, $y = -t$, $z = 4t$.
4. All vectors of the form $\begin{pmatrix} a, & b, & c \end{pmatrix}$, where $b = a + c$.

Question 12. In each part, find a basis for the given subspace of $\mathbb{R}^4$, and state its dimension.

1. All vectors of the form $\begin{pmatrix} a, & b, & c, & 0 \end{pmatrix}$.
2. All vectors of the form $\begin{pmatrix} a, & b, & c, & d \end{pmatrix}$, where $d = a + b$ and $c = a - b$.

c) All vectors of the form $\begin{pmatrix} a, & b, & c, & d \end{pmatrix}$, where $a = b = c = d$.

Question 13. Show that the matrices

$$M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad M_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

form a basis for $\mathcal{M}_{2\times 2}$.

Question 14. Find the dimension of each of the following vector spaces.

1. The vector space of all diagonal $n \times n$ matrices.
2. The vector space of all symmetric $n \times n$ matrices.
3. The vector space of all upper triangular $n \times n$ matrices.
4. The vector space of all lower triangular $n \times n$ matrices.

Question 15. Find a standard basis vector in $\mathbb{R}^3$ that can be added to the set $\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ to produce a basis for $\mathbb{R}^3$.

1. $\boldsymbol{v}_1 = \begin{pmatrix}-1, & 2, & 3\end{pmatrix}$, $\boldsymbol{v}_2 = \begin{pmatrix}1, & 2, & -2\end{pmatrix}$.
2. $\boldsymbol{v}_1 = \begin{pmatrix}1, & -1, & 0\end{pmatrix}$, $\boldsymbol{v}_2 = \begin{pmatrix}3, & 1, & -2\end{pmatrix}$.

Question 16. Find standard basis vectors for $\mathbb{R}^4$ that can be added to the set $\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ to produce a basis for $\mathbb{R}^4$.

$$v_1 = \begin{pmatrix}1, & -4, & 2, & -3\end{pmatrix}, \quad v_2 = \begin{pmatrix}-3, & 8, & -4, & 6\end{pmatrix}$$

Question 17. The vectors $\boldsymbol{v}_1 = \begin{pmatrix}1, & -2, & 3\end{pmatrix}$ and $\boldsymbol{v}_2 = \begin{pmatrix}0, & 5, & -3\end{pmatrix}$ are linearly independent. Enlarge the set $\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ to a basis for $\mathbb{R}^3$.

Question 18. The vectors $\boldsymbol{v}_1 = \begin{pmatrix}1, & 0, & 0, & 0\end{pmatrix}$ and $\boldsymbol{v}_2 = \begin{pmatrix}1, & 1, & 0, & 0\end{pmatrix}$ are linearly independent. Enlarge the set $\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ to a basis for $\mathbb{R}^4$.

Question 19. Consider the bases $B_1 = \{\boldsymbol{u}_1,\ \boldsymbol{u}_2\}$ and $B_2 = \{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ for $\mathbb{R}^2$, where

(a)

$$\boldsymbol{u}_1 = \begin{pmatrix}2\\2\end{pmatrix}, \quad \boldsymbol{u}_2 = \begin{pmatrix}4\\-1\end{pmatrix}, \quad \boldsymbol{v}_1 = \begin{pmatrix}1\\3\end{pmatrix}, \quad \boldsymbol{v}_2 = \begin{pmatrix}-1\\-1\end{pmatrix}$$

(b)

$$\boldsymbol{u}_1 = \begin{pmatrix}1\\0\end{pmatrix}, \quad \boldsymbol{u}_2 = \begin{pmatrix}0\\1\end{pmatrix}, \quad \boldsymbol{v}_1 = \begin{pmatrix}2\\1\end{pmatrix}, \quad \boldsymbol{v}_2 = \begin{pmatrix}-3\\4\end{pmatrix}$$

1. Find the transition matrix from B_2 to B_1

2. Find the transition matrix from B_1 to B_2
3. Compute the coordinate vector $[\boldsymbol{w}]_{B_1}$, where

$$\boldsymbol{w} = \begin{pmatrix} 3 \\ -5 \end{pmatrix}$$

and compute $[\boldsymbol{w}]_{B_2}$ using the transition matrix from B_1 to B_2

4. Compute $[\boldsymbol{w}]_{B_2}$ directly

Question 20. Consider the bases $B_1 = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ and $B_2 = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ for $\mathbb{R}^3$, where

(a)

$$\boldsymbol{u}_1 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \quad \boldsymbol{u}_2 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \quad \boldsymbol{u}_3 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \quad \boldsymbol{v}_1 = \begin{pmatrix} 3 \\ 1 \\ -5 \end{pmatrix}, \quad \boldsymbol{v}_2 = \begin{pmatrix} 1 \\ 1 \\ -3 \end{pmatrix}, \quad \boldsymbol{v}_3 = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$$

(b)

$$\boldsymbol{u}_1 = \begin{pmatrix} -3 \\ 0 \\ -3 \end{pmatrix}, \quad \boldsymbol{u}_2 = \begin{pmatrix} -3 \\ 2 \\ -1 \end{pmatrix}, \quad \boldsymbol{u}_3 = \begin{pmatrix} 1 \\ 6 \\ -1 \end{pmatrix}, \quad \boldsymbol{v}_1 = \begin{pmatrix} -6 \\ -6 \\ 0 \end{pmatrix}, \quad \boldsymbol{v}_2 = \begin{pmatrix} -2 \\ -6 \\ 4 \end{pmatrix}, \quad \boldsymbol{v}_3 = \begin{pmatrix} -2 \\ -3 \\ 7 \end{pmatrix}$$

1. Find the transition matrix from B_2 to B_1
2. Find the transition matrix from B_1 to B_2
3. Compute the coordinate vector $[\boldsymbol{w}]_{B_1}$, where

$$\boldsymbol{w} = \begin{pmatrix} -5 \\ 8 \\ -5 \end{pmatrix}$$

and compute $[\boldsymbol{w}]_{B_2}$ using the transition matrix from B_1 to B_2

4. Compute $[\boldsymbol{w}]_{B_2}$ directly

Question 21. Let $B_1 = \{\boldsymbol{u}_1, \boldsymbol{u}_2\}$ and $B_2 = \{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ be the bases for $\mathbb{R}^2$, where

$$\boldsymbol{u}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad \boldsymbol{u}_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \quad \boldsymbol{v}_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \quad \boldsymbol{v}_2 = \begin{pmatrix} 1 \\ 4 \end{pmatrix}.$$

An efficient way to compute the transition matrix $P_{B_1 \to B_2}$ is as follows

Step 1. Form the matrix $\left(B_2 \mid B_1\right)$

Step 2. Use elementary row operations to reduce the matrix in Step 1 to reduced row echelon form

Step 3. The resulting matrix will be $\left(I \mid P_{B_1 \to B_2}\right)$

Step 4. Extract the matrix $P_{B_1 \to B_2}$ from the right side of the matrix in Step 3

In diagram

$$\left(\text{new basis} \mid \text{old basis}\right) \xrightarrow{\text{row operations}} \left(I \mid \text{transition from old to new}\right) \tag{1}$$

1. Apply the above procedure to find the transition matrix $P_{B_2 \to B_1}$
2. Apply the above procedure to find the transition matrix $P_{B_1 \to B_2}$
3. Confirm that $P_{B_2 \to B_1}$ and $P_{B_1 \to B_2}$ are inverses of one another
4. Let $\boldsymbol{w} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Find $[\boldsymbol{w}]_{B_1}$ and then use the matrix $P_{B_1 \to B_2}$ to compute $[\boldsymbol{w}]_{B_2}$ from $[\boldsymbol{w}]_{B_1}$
5. Let $\boldsymbol{w} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$. Find $[\boldsymbol{w}]_{B_2}$ and then use the matrix $P_{B_2 \to B_1}$ to compute $[\boldsymbol{w}]_{B_1}$ from $[\boldsymbol{w}]_{B_2}$

Question 22. Let S be the standard basis for $\mathbb{R}^2$, and let $B = \{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ be the basis in which

$$\boldsymbol{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad \boldsymbol{v}_2 = \begin{pmatrix} -3 \\ 4 \end{pmatrix}$$

1. Find the transition matrix $P_{B \to S}$ by inspection
2. Use Formula (1) to find the transition matrix $P_{S \to B}$
3. Confirm that $P_{B \to S}$ and $P_{S \to B}$ are inverses of one another
4. Let $\boldsymbol{w} = \begin{pmatrix} 5 \\ -3 \end{pmatrix}$. Find $[\boldsymbol{w}]_B$ and then use the matrix $P_{B \to S}$ to compute $[\boldsymbol{w}]_S$ from $[\boldsymbol{w}]_B$
5. Let $\boldsymbol{w} = \begin{pmatrix} 3 \\ -5 \end{pmatrix}$. Find $[\boldsymbol{w}]_S$ and then use the matrix $P_{S \to B}$ to compute $[\boldsymbol{w}]_B$ from $[\boldsymbol{w}]_S$

Question 23. Let S be the standard basis for $\mathbb{R}^3$, and let $B = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ be the basis in which

$$\boldsymbol{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \quad \boldsymbol{v}_2 = \begin{pmatrix} 2 \\ 5 \\ 0 \end{pmatrix}, \quad \boldsymbol{v}_3 = \begin{pmatrix} 3 \\ 3 \\ 8 \end{pmatrix}$$

1. Find the transition matrix $P_{B \to S}$ by inspection
2. Use Formula (1) to find the transition matrix $P_{S \to B}$
3. Confirm that $P_{B \to S}$ and $P_{S \to B}$ are inverses of one another
4. Let $\boldsymbol{w} = \begin{pmatrix} 5 \\ -3 \\ 1 \end{pmatrix}$. Find $[\boldsymbol{w}]_B$ and then use the matrix $P_{B \to S}$ to compute $[\boldsymbol{w}]_S$ from $[\boldsymbol{w}]_B$
5. Let $\boldsymbol{w} = \begin{pmatrix} 3 \\ -5 \\ 0 \end{pmatrix}$. Find $[\boldsymbol{w}]_S$ and then use the matrix $P_{S \to B}$ to compute $[\boldsymbol{w}]_B$ from $[\boldsymbol{w}]_S$

Question 24. Let $S = \{\boldsymbol{e}_1, \boldsymbol{e}_2\}$ be the standard basis for $\mathbb{R}^2$, and let $B = \{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ be the basis that results when the vectors in S are reflected about the line $y = x$.

1. Find the transition matrix $P_{B \to S}$
2. Show that $P_{B \to S}^{\top} = P_{S \to B}$

Question 25. Let $S = \{\boldsymbol{e}_1, \boldsymbol{e}_2\}$ be the standard basis for $\mathbb{R}^2$, and let $B = \{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ be the basis that results when the vectors in S are reflected about the line that makes an angle θ with the positive $x-$axis.

1. Find the transition matrix $P_{B \to S}$
2. Show that $P_{B \to S}^{\top} = P_{S \to B}$

Question 26. Find the domain and codomain of the transformation $T_A(\boldsymbol{x}) = A\boldsymbol{x}$

1. $A \in \mathcal{M}_{3\times 2}$
2. $A \in \mathcal{M}_{2\times 3}$
3. $A \in \mathcal{M}_{3\times 3}$
4. $A \in \mathcal{M}_{1\times 6}$
5. $A \in \mathcal{M}_{4\times 5}$
6. $A \in \mathcal{M}_{5\times 4}$
7. $A \in \mathcal{M}_{4\times 4}$
8. $A \in \mathcal{M}_{3\times 1}$

Question 27. Find the domain and codomain of the transformation defined by the equations

1.
$$\begin{aligned} w_1 &= 4x_1 + 5x_2 \\ w_2 &= x_1 - 8x_2 \end{aligned}$$

2.
$$\begin{aligned} w_1 &= 5x_1 - 7x_2 \\ w_2 &= 6x_1 + x_2 \\ w_3 &= 2x_1 + 3x_2 \end{aligned}$$

3.
$$\begin{aligned} w_1 &= x_1 - 4x_2 + 8x_3 \\ w_2 &= -x_1 + 4x_2 + 2x_3 \\ w_3 &= -3x_1 + 2x_2 - 5x_3 \end{aligned}$$

4.
$$\begin{aligned} w_1 &= 2x_1 + 7x_2 - 4x_3 \\ w_2 &= 4x_1 - 3x_2 + 2x_3 \end{aligned}$$

Question 28. Find the standard matrix for the transformation defined below

1.

$$\begin{aligned} w_1 &= 2x_1 - 3x_2 + x_3 \\ w_2 &= 3x_1 + 5x_2 - x_3 \end{aligned}$$

2.

$$\begin{aligned} w_1 &= 7x_1 + 2x_2 - 8x_3 \\ w_2 &= -x_2 + 5x_3 \\ w_3 &= 4x_1 + 7x_2 - x_3 \end{aligned}$$

3. $T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -x_1 \\ x_1 + 3x_2 \\ x_1 - x_2 \end{pmatrix}$

4. $T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 7x_1 + 2x_2 - x_3 + x_4 \\ x_2 + x_3 \\ -x_1 \end{pmatrix}$

5. $T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

6. $T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_4 \\ x_1 \\ x_3 \\ x_2 \\ x_1 - x_3 \end{pmatrix}$

Question 29. Find $T_A(\boldsymbol{x})$.

1. $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, $\boldsymbol{x} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$

2. $A = \begin{pmatrix} -1 & 2 & 0 \\ 3 & 1 & 5 \end{pmatrix}$, $\boldsymbol{x} = \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix}$

3. $A = \begin{pmatrix} -2 & 1 & 4 \\ 3 & 5 & 7 \\ 6 & 0 & -1 \end{pmatrix}$, $\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

4. $A = \begin{pmatrix} -1 & 1 \\ 2 & 4 \\ 7 & 8 \end{pmatrix}$, $\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

Question 30. The images of the standard basis vectors for $\mathbb{R}^3$ are given for a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$. Find the standard matrix for the transformation, and find $T(\boldsymbol{x})$.

1. $T(\boldsymbol{e}_1) = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}$, $T(\boldsymbol{e}_2) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, $T(\boldsymbol{e}_3) = \begin{pmatrix} 4 \\ -3 \\ -1 \end{pmatrix}$, $\boldsymbol{x} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$
2. $T(\boldsymbol{e}_1) = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$, $T(\boldsymbol{e}_2) = \begin{pmatrix} -3 \\ -1 \\ 0 \end{pmatrix}$, $T(\boldsymbol{e}_3) = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$, $\boldsymbol{x} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$

Question 31. Use matrix multiplication to find the reflection of $\begin{pmatrix} -1, & 2 \end{pmatrix}$ about the

1. $x-$axis 2. $y-$axis 3. line $y = x$

Question 32. Use matrix multiplication to find the reflection of $\begin{pmatrix} a, & b \end{pmatrix}$ about the

1. $x-$axis 2. $y-$axis 3. line $y = x$

Question 33. Use matrix multiplication to find the reflection of $\begin{pmatrix} 2, & -5, & 3 \end{pmatrix}$ about the

1. $xy-$plane 2. $xz-$plane 3. $yz-$plane

Question 34. Use matrix multiplication to find the reflection of $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ about the

1. xy-plane 2. xz-plane 3. yz-plane

Question 35. Use matrix multiplication to find the orthogonal projection of $\begin{pmatrix} 2 \\ -5 \end{pmatrix}$ onto the

1. x-axis 2. y-axis

Question 36. Use matrix multiplication to find the orthogonal projection of $\begin{pmatrix} a \\ b \end{pmatrix}$ onto the

1. x-axis 2. y-axis

Question 37. Use matrix multiplication to find the orthogonal projection of $\begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix}$ onto the

1. xy-plane 2. xz-plane 3. yz-plane

Question 38. Use matrix multiplication to find the orthogonal projection of $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ onto the

1. xy-plane 2. xz-plane 3. yz-plane

Question 39. Use matrix multiplication to find the image of the vector $\begin{pmatrix} 3 \\ -4 \end{pmatrix}$ when it is rotated about the origin through an angle of

1. $\theta = 30°$ 2. $\theta = -60°$ 3. $\theta = 45°$ 4. $\theta = 90°$

Question 40. Use matrix multiplication to find the image of the nonzero vector $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ when it is rotated about the origin through

1. a positive angle θ 2. a negative angle $-\theta$

Question 41. Use matrix multiplication to find the image of the vector $\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$ if it is rotated

1. $30°$ clockwise about the positive x-axis.
2. $30°$ counterclockwise about the positive y-axis.
3. $45°$ clockwise about the positive y-axis.
4. $90°$ counterclockwise about the positive z-axis.

Question 42. Use matrix multiplication to find:

1. The contraction of $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ with factor $\alpha = \frac{1}{2}$.
2. The dilation of $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ with factor $\alpha = 3$.

Question 43. Use matrix multiplication to find:

1. The contraction of $\begin{pmatrix} a \\ b \end{pmatrix}$ with factor $\frac{1}{\alpha}$, where $\alpha > 1$.
2. The dilation of $\begin{pmatrix} a \\ b \end{pmatrix}$ with factor α, where $\alpha > 1$.

Question 44. Use matrix multiplication to find:

1. The contraction of $\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$ with factor $\frac{1}{4}$.
2. The dilation of $\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$ with factor 2.

Question 45. Use matrix multiplication to find:

1. The contraction of $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ with factor $\frac{1}{\alpha}$, where $\alpha > 1$.
2. The dilation of $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ with factor α, where $\alpha > 1$.

Question 46. Use matrix multiplication to find:

1. The compression of $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ in the x-direction with factor $\frac{1}{2}$.
2. The compression of $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ in the y-direction with factor $\frac{1}{2}$.

Question 47. Use matrix multiplication to find:

1. The expansion of $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ in the x-direction with factor 3.
2. The expansion of $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ in the y-direction with factor 3.

Question 48. Use matrix multiplication to find:

1. The compression of $\begin{pmatrix} a \\ b \end{pmatrix}$ in the x-direction with factor $\frac{1}{\alpha}$, where $\alpha > 1$.
2. The expansion of $\begin{pmatrix} a \\ b \end{pmatrix}$ in the y-direction with factor α, where $\alpha > 1$.

Question 49. In each part, determine whether the operators T_1 and T_2 commute, i.e. whether $T_1 \circ T_2 = T_2 \circ T_1$.

1. $T_1 : \mathbb{R}^2 \to \mathbb{R}^2$ is the reflection about the line $y = x$, and $T_2 : \mathbb{R}^2 \to \mathbb{R}^2$ is the orthogonal projection onto the x-axis.
2. $T_1 : \mathbb{R}^2 \to \mathbb{R}^2$ is the reflection about the x-axis, and $T_2 : \mathbb{R}^2 \to \mathbb{R}^2$ is the reflection about the line $y = x$.
3. $T_1 : \mathbb{R}^2 \to \mathbb{R}^2$ is the orthogonal projection onto the x-axis, and $T_2 : \mathbb{R}^2 \to \mathbb{R}^2$ is the orthogonal projection onto the y-axis.
4. $T_1 : \mathbb{R}^2 \to \mathbb{R}^2$ is the rotation about the origin through an angle of $\frac{\pi}{4}$, and $T_2 : \mathbb{R}^2 \to \mathbb{R}^2$ is the reflection about the y-axis.
5. $T_1 : \mathbb{R}^3 \to \mathbb{R}^3$ is a dilation with factor α, and $T_2 : \mathbb{R}^3 \to \mathbb{R}^3$ is a contraction with factor $\frac{1}{\alpha}$, where $\alpha > 1$.
6. $T_1 : \mathbb{R}^3 \to \mathbb{R}^3$ is the rotation about the x-axis through an angle θ_1, and $T_2 : \mathbb{R}^3 \to \mathbb{R}^3$ is the rotation about the z-axis through an angle θ_2.

Question 50. Find the standard matrix for the stated composition in $\mathbb{R}^2$.

1. A rotation of 90°, followed by a reflection about the line $y = x$.
2. An orthogonal projection onto the y-axis, followed by a contraction with factor $\frac{1}{2}$.
3. A reflection about the x-axis, followed by a dilation with factor 3, followed by a rotation about the origin of 60°.
4. A rotation about the origin of 60°, followed by an orthogonal projection onto the x-axis, followed by a reflection about the line $y = x$.
5. A dilation with factor 2, followed by a rotation about the origin of 45°, followed by a reflection about the y-axis.
6. A rotation about the origin of 15°, followed by a rotation about the origin of 105°, followed by a rotation about the origin of 60°.

Question 51. Find the standard matrix for the stated composition in $\mathbb{R}^3$.

1. A reflection about the yz-plane, followed by an orthogonal projection onto the xz-plane.
2. A rotation of $45°$ about the y-axis, followed by a dilation with factor $\sqrt{2}$.
3. An orthogonal projection onto the xy-plane, followed by a reflection about the yz-plane.
4. A rotation of $30°$ about the x-axis, followed by a rotation of $30°$ about the z-axis, followed by a contraction with factor $\frac{1}{4}$.
5. A reflection about the xy-plane, followed by a reflection about the xz-plane, followed by an orthogonal projection onto the yz-plane.
6. A rotation of $270°$ about the x-axis, followed by a rotation of $90°$ about the y-axis, followed by a rotation of $180°$ about the z-axis.

Question 52. Let $\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ be a vector in $\mathbb{R}^2$. Consider the linear transformations $T_1 : \mathbb{R}^2 \to \mathbb{R}^2$ and $T_2 : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$T_1(\boldsymbol{x}) = \begin{pmatrix} x_1 + x_2 \\ x_1 - x_2 \end{pmatrix}, \quad T_2(\boldsymbol{x}) = \begin{pmatrix} 3x_1 \\ 2x_1 + 4x_2 \end{pmatrix}$$

1. Find the standard matrices for T_1 and T_2.
2. Find the standard matrices for $T_1 \circ T_2$ and $T_2 \circ T_1$.
3. Find the standard matrices for $T_1 \circ T_2 \circ T_1$ and $T_1 \circ T_2 \circ T_2$.
4. Use the matrices obtained in part 2 to find formulas for $T_1\left(T_2\left(\boldsymbol{x}\right)\right)$ and $T_2\left(T_1\left(\boldsymbol{x}\right)\right)$

Question 53. Let $\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be a vector in $\mathbb{R}^3$. Consider the linear transformations $T_1 : \mathbb{R}^3 \to \mathbb{R}^3$ and $T_2 : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$T_1(\boldsymbol{x}) = \begin{pmatrix} 4x_1 \\ -2x_1 + x_2 \\ -x_1 - 3x_3 \end{pmatrix}, \quad T_2(\boldsymbol{x}) = \begin{pmatrix} x_1 + 2x_2 \\ 2x_3 \\ 4x_1 - x_3 \end{pmatrix}$$

1. Find the standard matrices for T_1 and T_2.
2. Find the standard matrices for $T_1 \circ T_2$ and $T_2 \circ T_1$.
3. Find the standard matrices for $T_1 \circ T_2 \circ T_1$ and $T_1 \circ T_2 \circ T_2$.
4. Use the matrices obtained in part 2 to find formulas for $T_1\left(T_2\left(\boldsymbol{x}\right)\right)$ and $T_2\left(T_1\left(\boldsymbol{x}\right)\right)$