Tutorial 5

Vector spaces and linear independence

Question 1. Consider $\mathbb{R}^2$ together with addition and scalar multiplication defined as follows: for any $\boldsymbol{u} = \begin{pmatrix} u_1, & u_2 \end{pmatrix}$, $\boldsymbol{v} = \begin{pmatrix} v_1, & v_2 \end{pmatrix} \in \mathbb{R}^2$, and any $\alpha \in \mathbb{R}$

$$\boldsymbol{u} + \boldsymbol{v} = \begin{pmatrix} u_1 + v_1, & u_2 + v_2 \end{pmatrix}, \quad \alpha \otimes \boldsymbol{u} = \begin{pmatrix} 0, & \alpha u_2 \end{pmatrix}$$

1. Compute $\boldsymbol{u} + \boldsymbol{v}$ and $\alpha \otimes \boldsymbol{u}$ for $\boldsymbol{u} = \begin{pmatrix} -1, & 2 \end{pmatrix}$, $\boldsymbol{v} = \begin{pmatrix} 3, & 4 \end{pmatrix}$ and $\alpha = 3$.
2. Prove that $(\mathbb{R}^2, +, \otimes)$ is closed under addition and scalar multiplication.
3. Since vector addition in $(\mathbb{R}^2, +, \otimes)$ coincides with standard vector addition in the usual vector space $(\mathbb{R}^2, +, \cdot)$, certain vector space axioms must hold for $(\mathbb{R}^2, +, \otimes)$ because they are known to hold in $(\mathbb{R}^2, +, \cdot)$. Identify which axioms these are.
4. Show that Axioms 5, 6, 7 of a vector space hold in $(\mathbb{R}^2, +, \otimes)$.
5. Show that Axiom 8 does not hold and hence that $(\mathbb{R}^2, +, \otimes)$ is not a vector space

Question 2. Consider $\mathbb{R}^2$ together with addition and scalar multiplication defined as follows: for any $\boldsymbol{u} = \begin{pmatrix} u_1, & u_2 \end{pmatrix}$, $\boldsymbol{v} = \begin{pmatrix} v_1, & v_2 \end{pmatrix} \in \mathbb{R}^2$, and any $\alpha \in \mathbb{R}$

$$\boldsymbol{u} \oplus \boldsymbol{v} = \begin{pmatrix} u_1 + v_1 + 1, & u_2 + v_2 + 1 \end{pmatrix}, \quad \alpha\boldsymbol{u} = \begin{pmatrix} \alpha u_1, & \alpha u_2 \end{pmatrix}$$

1. Compute $\boldsymbol{u} \oplus \boldsymbol{v}$ and $\alpha\boldsymbol{u}$ for $\boldsymbol{u} = \begin{pmatrix} 0, & 4 \end{pmatrix}$, $\boldsymbol{v} = \begin{pmatrix} 1, & -3 \end{pmatrix}$ and $\alpha = 2$.
2. Show that $\begin{pmatrix} 0, & 0 \end{pmatrix}$ does not serve as the additive identity in $(\mathbb{R}^2, \oplus, \cdot)$.
3. Prove that the additive identity in $(\mathbb{R}^2, \oplus, \cdot)$ is given by $\begin{pmatrix} -1, & -1 \end{pmatrix}$
4. Show that Axiom 4 holds by finding the additive inverse of any given $\boldsymbol{u} \in \mathbb{R}^2$
5. Identify two vector space axioms that do not hold in $(\mathbb{R}^2, \oplus, \cdot)$.

Question 3. For each of the following sets equipped with the given operations, determine whether it forms a vector space. For those that are not vector spaces identify the vector space axioms that fail.

1. The set of all real numbers with the standard operations of addition and multiplication.
2. The set of all pairs of real numbers of the form $\begin{pmatrix} x, & 0 \end{pmatrix}$ with the standard vector addition and scalar multiplication in $\mathbb{R}^2$.
3. The set of all pairs of real numbers of the form $\begin{pmatrix} x, & y \end{pmatrix}$ such that $x \geq 0$, with the standard vector addition and scalar multiplication in $\mathbb{R}^2$.
4. The set of all $n-$tuples of real numbers that have the form $\begin{pmatrix} x, & x, & \dots, & x \end{pmatrix}$ with the standard vector addition and scalar multiplication in $\mathbb{R}^n$.

5. The set $\mathbb{R}^3$ with the standard vector addition, but with scalar multiplication defined as

$$\alpha \otimes \left(u_1, \quad u_2, \quad u_3\right) = \left(\alpha^2 u_1, \quad \alpha^2 u_2, \quad \alpha^2 u_3\right).$$

6. The set of all invertible 2×2 matrices, together with the standard matrix addition and scalar multiplication.

7. The set of all diagonal 2×2 matrices of the form

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix},$$

   together with the standard matrix addition and scalar multiplication

8. The set of all real-valued functions f defined everywhere on the real line satisfying the condition $f(1) = 0$, together with addition and scalar multiplication defined as follows

$$(f + g)(x) = f(x) + g(x), \quad (\alpha f)(x) = \alpha f(x)$$

9. The subset of $\mathbb{R}^2$ consisting of all pairs of the form $\left(1, \quad y\right)$ with the operations

$$\left(1, \quad y\right) \oplus \left(1, \quad y'\right) = \left(1, \quad y + y'\right), \quad \alpha \otimes \left(1, \quad y\right) = \left(1, \quad \alpha y\right)$$

10. The set of polynomials of the form $a_0 + a_1 x$ with the operations

$$(a_0 + a_1 x) + (b_0 + b_1 x) = (a_0 + b_0) + (a_1 + b_1)x$$

   and

$$\alpha(a_0 + a_1 x) = \alpha a_0 + \alpha a_1 x$$

Question 4. Verify Axioms 1, 2, 5, 6, and 7 for the vector space $\mathcal{M}_{2\times 2}$.

Question 5. Verify Axioms 2, 5, 6, 7, and 8 for the vector space $\mathcal{F}(\mathbb{R}, \mathbb{R})$.

Question 6. Show that $\mathbb{R}^2$ with the usual addition and scalar multiplication defined as

$$\alpha \left(u_1, \quad u_2\right) = \left(\alpha u_1, \quad 0\right)$$

satisfy Axioms 1-7.

Question 7. Consider $\mathbb{R}_{>0}$, the set of positive real numbers. Define addition and scalar multiplication as follows: for any $u, v \in \mathbb{R}_{>0}$ and any $\alpha \in \mathbb{R}$

$$u \oplus v = uv, \quad \alpha \otimes u = u^\alpha$$

Verify that Axioms 1 – 5, 7, and 8 hold.

Question 8. Show that the set of all points in $\mathbb{R}^2$ lying on a line is a subspace of $(\mathbb{R}^2, +, \cdot)$ iff the line passes through the origin.

Question 9. Show that the set of all points in $\mathbb{R}^3$ lying in a plane is a subspace of $(\mathbb{R}^3, +, \cdot)$ iff the plane passes through the origin.

$\Longrightarrow$ If a plane is a subspace of $(\mathbb{R}^3, +, \cdot)$, by definition it contains the zero vector $\begin{pmatrix} 0, & 0, & 0 \end{pmatrix}$, which is the origin.

$\Longleftarrow$ If a plane passes through the origin, it can be represented by an equation of the form

$$ax + by + cz = 0$$

for some $a, b, c \in \mathbb{R}$. Let

$$W = \left\{ \begin{pmatrix} x, & y, & z \end{pmatrix} \;\middle|\; ax + by + cz = 0 \right\}$$

denote such a plane. Take any $\begin{pmatrix} x_1, & y_1, & z_1 \end{pmatrix}$, $\begin{pmatrix} x_2, & y_2, & z_2 \end{pmatrix}$ from W and any $\alpha \in \mathbb{R}$. Then

$$ax_1 + by_1 + cz_1 = 0, \quad ax_2 + by_2 + cz_2 = 0.$$

We have

$$\begin{pmatrix} x_1, & y_1, & z_1 \end{pmatrix} + \begin{pmatrix} x_2, & y_2, & z_1 \end{pmatrix} = \begin{pmatrix} x_1 + x_2, & y_1 + y_2, & z_1 + z_2 \end{pmatrix}.$$

We have

$$a(x_1 + x_2) + b(y_1 + y_2) + c(z_1 + z_2) = (ax_1 + by_1 + cz_1) + (ax_2 + by_2 + cz_2) = 0$$

$$\Longrightarrow \begin{pmatrix} x_1, & y_1, & z_1 \end{pmatrix} + \begin{pmatrix} x_2, & y_2, & z_2 \end{pmatrix} \in W.$$

Thus W is closed under addition. Furthermore, W is also closed under scalar multiplication:

$$\alpha \begin{pmatrix} x_1, & y_1, & z_1 \end{pmatrix} = \begin{pmatrix} \alpha x_1, & \alpha y_1, & \alpha z_1 \end{pmatrix}, \quad a(\alpha x_1) + b(\alpha y_1) + c(\alpha z_1) = \alpha(ax_1 + by_1 + cz_1) = 0$$

$$\Longrightarrow \alpha \begin{pmatrix} x_1, & y_1, & z_1 \end{pmatrix} \in W.$$

Question 10. Determine which of the following are subspaces of $\mathbb{R}^3$.

1. All vectors of the form $\begin{pmatrix} a, & 0, & 0 \end{pmatrix}$
2. All vectors of the form $\begin{pmatrix} a, & 1, & 1 \end{pmatrix}$
3. All vectors of the form $\begin{pmatrix} a, & b, & c \end{pmatrix}$, where $b = a + c$
4. All vectors of the form $\begin{pmatrix} a, & b, & c \end{pmatrix}$, where $b = a + c + 1$
5. All vectors of the form $\begin{pmatrix} a, & b, & 0 \end{pmatrix}$

Question 11. Determine which of the following are subspaces of $\mathcal{M}_{n\times n}$.

1. The set of all diagonal $n \times n$ matrices
2. The set of all $n \times n$ matrices A such that $\det(A) = 0$
3. The set of all $n \times n$ matrices A such that $\operatorname{tr}(A) = 0$
4. The set of all symmetric $n \times n$ matrices
5. The set of all $n \times n$ matrices A such that $A^\top = -A$
6. The set of all $n \times n$ matrices A for which $A\boldsymbol{x} = \mathbf{0}$ has only the trivial solution
7. The set of all $n \times n$ matrices A such that $AB = BA$ for some fixed $n \times n$ matrix B.

Question 12. Which of the following are subspaces of $\mathbb{R}^\infty$?

1. All sequences $\boldsymbol{v} \in \mathbb{R}^\infty$ of the form $\boldsymbol{v} = \begin{pmatrix} v, & 0, & v, & 0, & v, & 0, & \ldots \end{pmatrix}$.
2. All sequences $\boldsymbol{v} \in \mathbb{R}^\infty$ of the form $\boldsymbol{v} = \begin{pmatrix} v, & 1, & v, & 1, & v, & 1, & \ldots \end{pmatrix}$.
3. All sequences $\boldsymbol{v} \in \mathbb{R}^\infty$ of the form $\boldsymbol{v} = \begin{pmatrix} v, & 2v, & 4v, & 8v, & 16v, & \ldots \end{pmatrix}$.
4. All sequences in $\mathbb{R}^\infty$ whose components are 0 from some point on.

Question 13. Which of the following are linear combinations of $\boldsymbol{u} = \begin{pmatrix} 0, & -2, & 2 \end{pmatrix}$, $\boldsymbol{v} = \begin{pmatrix} 1, & 3, & -1 \end{pmatrix}$

1. $\begin{pmatrix} 2, & 2, & 2 \end{pmatrix}$ 2. $\begin{pmatrix} 0, & 4, & 5 \end{pmatrix}$ 3. $\begin{pmatrix} 0, & 0, & 0 \end{pmatrix}$

Question 14. Express the following as linear combinations of $\boldsymbol{u} = \begin{pmatrix} 2, & 1, & 4 \end{pmatrix}$, $\boldsymbol{v} = \begin{pmatrix} 1, & -2, & 3 \end{pmatrix}$ and $\boldsymbol{w} = \begin{pmatrix} 3, & 2, & 5 \end{pmatrix}$.

1. $\begin{pmatrix} -9, & -7, & -15 \end{pmatrix}$ 2. $\begin{pmatrix} 6, & 11, & 6 \end{pmatrix}$ 3. $\begin{pmatrix} 0, & 0, & 0 \end{pmatrix}$

Question 15. Which of the following are linear combinations of

$$A = \begin{pmatrix} 4 & 0 \\ -2 & -2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 2 \\ 1 & 4 \end{pmatrix}$$

1. $\begin{pmatrix} 6 & -8 \\ -1 & -8 \end{pmatrix}$ 2. $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ 3. $\begin{pmatrix} -1 & 5 \\ 7 & 1 \end{pmatrix}$

Question 16. In each part, determine whether the vectors span $\mathbb{R}^3$.

1. $\boldsymbol{v}_1 = \begin{pmatrix}2, & 2, & 2\end{pmatrix}$, $\boldsymbol{v}_2 = \begin{pmatrix}0, & 0, & 3\end{pmatrix}$, $\boldsymbol{v}_3 = \begin{pmatrix}0, & 1, & 1\end{pmatrix}$
2. $\boldsymbol{v}_1 = \begin{pmatrix}2, & -1, & 3\end{pmatrix}$, $\boldsymbol{v}_2 = \begin{pmatrix}4, & 1, & 2\end{pmatrix}$, $\boldsymbol{v}_3 = \begin{pmatrix}8, & -1, & 8\end{pmatrix}$

Question 17. Suppose that $\boldsymbol{v}_1 = \begin{pmatrix}2, & 1, & 0, & 3\end{pmatrix}$, $\boldsymbol{v}_2 = \begin{pmatrix}3, & -1, & 5, & 2\end{pmatrix}$, and $\boldsymbol{v}_3 = \begin{pmatrix}-1, & 0, & 2, & 1\end{pmatrix}$. Which of the following vectors are in $\text{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$?

1. $\begin{pmatrix}2, & 3, & -7, & 3\end{pmatrix}$
2. $\begin{pmatrix}0, & 0, & 0, & 0\end{pmatrix}$
3. $\begin{pmatrix}1, & 1, & 1, & 1\end{pmatrix}$
4. $\begin{pmatrix}-4, & 6, & -13, & 4\end{pmatrix}$

Question 18. Determine whether the solution space of the system $A\boldsymbol{x} = \mathbf{0}$ is a line through the origin, a plane through the origin, or the origin only. If it is a plane, find an equation for it. If it is a line, find parametric equations for it.

1. $A = \begin{pmatrix}-1 & 1 & 1\\ 3 & -1 & 0\\ 2 & -4 & -5\end{pmatrix}$
2. $A = \begin{pmatrix}1 & 2 & 3\\ 2 & 5 & 3\\ 1 & 0 & 8\end{pmatrix}$
3. $A = \begin{pmatrix}1 & -3 & 1\\ 2 & -6 & 2\\ 3 & -9 & 3\end{pmatrix}$
4. $A = \begin{pmatrix}1 & -1 & 1\\ 2 & -1 & 4\\ 3 & 1 & 11\end{pmatrix}$
5. $A = \begin{pmatrix}10 & 4 & 21\\ 0 & -4 & 3\\ -5 & -1 & -12\end{pmatrix}$
6. $A = \begin{pmatrix}18 & -9 & -14\\ 6 & -3 & -5\\ -3 & 1 & 2\end{pmatrix}$
7. $A = \begin{pmatrix}3 & -6 & 9\\ -2 & 7 & -2\\ 0 & 1 & 5\end{pmatrix}$
8. $A = \begin{pmatrix}3 & 6 & -9\\ 0 & 0 & -2\\ -2 & 1 & 5\end{pmatrix}$
9. $A = \begin{pmatrix}a & b & b\\ b & a & b\\ b & b & a\end{pmatrix}$, $a \neq 0$ or $b \neq 0$

Question 19. Explain why the following form linearly dependent sets of vectors

1. $\boldsymbol{u}_1 = \begin{pmatrix}-1, & 2, & 4\end{pmatrix}$, $\boldsymbol{u}_2 = \begin{pmatrix}5, & -10, & -20\end{pmatrix}$ in $\mathbb{R}^3$
2. $\boldsymbol{u}_1 = \begin{pmatrix}3, & -1\end{pmatrix}$, $\boldsymbol{u}_2 = \begin{pmatrix}4, & 5\end{pmatrix}$, $\boldsymbol{u}_3 = \begin{pmatrix}-2, & 7\end{pmatrix}$ in $\mathbb{R}^2$
3. $A = \begin{pmatrix}-3 & 4\\ 2 & 0\end{pmatrix}$, $B = \begin{pmatrix}3 & -4\\ -2 & 0\end{pmatrix}$ in $\mathcal{M}_{2\times 2}$

Question 20. In each part, determine whether the vectors are linearly independent or are linearly dependent in $\mathbb{R}^3$.

1. $\begin{pmatrix}-3, & 0, & 4\end{pmatrix}$, $\begin{pmatrix}5, & -1, & 2\end{pmatrix}$, $\begin{pmatrix}1, & 1, & 3\end{pmatrix}$
2. $\begin{pmatrix}-2, & 0, & 1\end{pmatrix}$, $\begin{pmatrix}3, & 2, & 5\end{pmatrix}$, $\begin{pmatrix}6, & -1, & 1\end{pmatrix}$, $\begin{pmatrix}7, & 0, & 2\end{pmatrix}$

Question 21. In each part, determine whether the vectors are linearly independent or are linearly dependent in $\mathbb{R}^4$.

1. $\begin{pmatrix}3, & 8, & 7, & -3\end{pmatrix}, \begin{pmatrix}1, & 5, & 3, & -1\end{pmatrix}, \begin{pmatrix}2, & -1, & 2, & 6\end{pmatrix}, \begin{pmatrix}4, & 2, & 6, & 4\end{pmatrix}$
2. $\begin{pmatrix}3, & 0, & -3, & 6\end{pmatrix}, \begin{pmatrix}0, & 2, & 3, & 1\end{pmatrix}, \begin{pmatrix}0, & -2, & -2, & 0\end{pmatrix}, \begin{pmatrix}-2, & 1, & 2, & 1\end{pmatrix}$

Question 22. Prove the following theorem

Theorem 1 *$S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n\}$ spans $\mathbb{R}^n$ iff the determinant*

$$\begin{vmatrix}\boldsymbol{v}_1\\ \boldsymbol{v}_2\\ \vdots\\ \boldsymbol{v}_n\end{vmatrix} \neq 0.$$

Question 23. In each part, determine whether the matrices are linearly independent or dependent.

1. $\begin{pmatrix}1 & 0\\ 1 & 2\end{pmatrix}, \begin{pmatrix}1 & 2\\ 2 & 1\end{pmatrix}, \begin{pmatrix}0 & 1\\ 2 & 1\end{pmatrix}$ in $\mathcal{M}_{2\times 2}$
2. $\begin{pmatrix}1 & -1\\ 0 & 0\end{pmatrix}, \begin{pmatrix}1 & -1\\ -2 & -2\end{pmatrix}, \begin{pmatrix}1 & 1\\ -2 & 2\end{pmatrix}$ in $\mathcal{M}_{2\times 2}$
3. $\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0 & 0\\ 0 & 1 & 0\end{pmatrix}, \begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}$ in $\mathcal{M}_{2\times 3}$.

Question 24. Determine all values of a for which the following matrices are linearly independent in $\mathcal{M}_{2\times 2}$

$$\begin{pmatrix}1 & 0\\ 1 & a\end{pmatrix}, \begin{pmatrix}-1 & 0\\ a & 1\end{pmatrix}, \begin{pmatrix}2 & 0\\ 1 & 3\end{pmatrix}$$

Question 25. In each part, determine whether the three vectors lie in a plane in $\mathbb{R}^3$

1. $\boldsymbol{v}_1 = \begin{pmatrix}2, & -2, & 0\end{pmatrix}$, $\boldsymbol{v}_2 = \begin{pmatrix}6, & 1, & 4\end{pmatrix}$, $\boldsymbol{v}_3 = \begin{pmatrix}2, & 0, & -4\end{pmatrix}$
2. $\boldsymbol{v}_1 = \begin{pmatrix}-6, & 7, & 2\end{pmatrix}$, $\boldsymbol{v}_2 = \begin{pmatrix}3, & 2, & 4\end{pmatrix}$, $\boldsymbol{v}_3 = \begin{pmatrix}4, & -1, & 2\end{pmatrix}$

Question 26. In each part, determine whether the three vectors lie on the same line in $\mathbb{R}^3$

1. $\boldsymbol{v}_1 = \begin{pmatrix}-1, & 2, & 3\end{pmatrix}$, $\boldsymbol{v}_2 = \begin{pmatrix}-2, & -4, & -6\end{pmatrix}$, $\boldsymbol{v}_3 = \begin{pmatrix}-3, & 6, & 0\end{pmatrix}$
2. $\boldsymbol{v}_1 = \begin{pmatrix}2, & -1, & 4\end{pmatrix}$, $\boldsymbol{v}_2 = \begin{pmatrix}4, & 2, & 3\end{pmatrix}$, $\boldsymbol{v}_3 = \begin{pmatrix}2, & 7, & -6\end{pmatrix}$
3. $\boldsymbol{v}_1 = \begin{pmatrix}4, & 6, & 8\end{pmatrix}$, $\boldsymbol{v}_2 = \begin{pmatrix}2, & 3, & 4\end{pmatrix}$, $\boldsymbol{v}_3 = \begin{pmatrix}-2, & -3, & -4\end{pmatrix}$

Question 27. For which values of λ do the following vectors form a linearly dependent set in $\mathbb{R}^3$?

$$\boldsymbol{v}_1 = \begin{pmatrix}\lambda, & -\frac{1}{2}, & -\frac{1}{2}\end{pmatrix}, \quad \boldsymbol{v}_2 = \begin{pmatrix}-\frac{1}{2}, & \lambda, & -\frac{1}{2}\end{pmatrix}, \quad \boldsymbol{v}_3 = \begin{pmatrix}-\frac{1}{2}, & -\frac{1}{2}, & \lambda\end{pmatrix}.$$

Question 28. For each part, first show that the vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ are linearly dependent in $\mathbb{R}^4$. Subsequently, demonstrate that each vector can be expressed as a linear combination of the remaining two.

1. $\boldsymbol{v}_1 = \begin{pmatrix}0, & 3, & 1, & -1\end{pmatrix}$, $\boldsymbol{v}_2 = \begin{pmatrix}6, & 0, & 5, & 1\end{pmatrix}$, $\boldsymbol{v}_3 = \begin{pmatrix}4, & -7, & 1, & 3\end{pmatrix}$
2. $\boldsymbol{v}_1 = \begin{pmatrix}1, & 2, & 3, & 4\end{pmatrix}$, $\boldsymbol{v}_2 = \begin{pmatrix}0, & 1, & 0, & -1\end{pmatrix}$, $\boldsymbol{v}_3 = \begin{pmatrix}1, & 3, & 3, & 3\end{pmatrix}$