Tutorial 2

Vectors and Matrices

1 Vectors

Question 1. Given points A, B, C, find coordinates of B such that

a $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are orthogonal

b $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are parallel

1. $A = (3, 5),\ B = (2, y),\ C = (2, 8)$
2. $A = (-2, 5),\ B = (1, y),\ C = (4, -3)$
3. $A = (1, 5),\ B = (-1, y),\ C = (2, -3)$
4. $A = (2, 1),\ B = (x, -2),\ C = (1, 3)$

Question 2. Consider the triangle with vertices A, B, and C, $\triangle ABC$. Define the vectors

$$\boldsymbol{a} = \overrightarrow{BC}, \quad \boldsymbol{b} = \overrightarrow{AC}, \quad \boldsymbol{c} = \overrightarrow{AB}.$$

Let M, N, and P denote the midpoints of the sides BC, AC, and AB, respectively.

Determine the expressions for the vectors $\overrightarrow{AM}$, $\overrightarrow{BN}$, and $\overrightarrow{CP}$ in terms of $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$. Subsequently, rewrite these expressions using only the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$.

Question 3. Given points A, B, find x such that the norm of the vector $\overrightarrow{AB}$, $\|\overrightarrow{AB}\|$ equals the specified value d.

1. $A = (2, -3),\ B = (x, 0),\ d = 5$
2. $A = (1, 4),\ B = (x, 1),\ d = 4$
3. $A = (1, 5),\ B = (1, x),\ d = 6$

Question 4. Prove that the triangle $\triangle ABC$ is isosceles.

1. $A = (-3, -2)$, $B = (1, 4)$, $C = (-5, 0)$.
2. $A = (7, -3, 6)$, $B = (11, -5, 3)$, $C = (10, -7, 8)$.

Question 5. Given vectors $\boldsymbol{a} = \begin{bmatrix} 3, & -2 \end{bmatrix}$ and $\boldsymbol{b} = \begin{bmatrix} -1, & 5 \end{bmatrix}$, find vector $\boldsymbol{c}$ such that:

- $\boldsymbol{a} \cdot \boldsymbol{c} = 17$.
- $\boldsymbol{b} \cdot \boldsymbol{c} = 3$.

Question 6. Find vectors $\boldsymbol{a}$ such that it is orthogonal to the given vector $\boldsymbol{b}$ and a norm equal to the specified value d.

1. $\boldsymbol{b} = \begin{bmatrix} 3, & 4 \end{bmatrix}$, $d = 15$
2. $\boldsymbol{b} = \begin{bmatrix} -3, & 2 \end{bmatrix}$, $d = 10$
3. $\boldsymbol{b} = \begin{bmatrix} 1, & 4 \end{bmatrix}$, $d = 8$
4. $\boldsymbol{b} = \begin{bmatrix} -1, & 2 \end{bmatrix}$, $d = 7$
5. $\boldsymbol{b} = \begin{bmatrix} 2, & -5 \end{bmatrix}$, $d = 21$

Question 7. Find the vector $\boldsymbol{c}$.

1. From the equation:
$$2\boldsymbol{c} + 3\boldsymbol{a} = \boldsymbol{b}, \quad \text{where} \quad \boldsymbol{a} = \begin{bmatrix} -1, & 2 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 0, & -2 \end{bmatrix}.$$
2. From the equation:
$$3\boldsymbol{c} - \boldsymbol{a} = 2\boldsymbol{b}, \quad \text{where} \quad \boldsymbol{a} = \begin{bmatrix} 3, & -1 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 0, & 2 \end{bmatrix}.$$
3. From the equation:
$$2\boldsymbol{c} - 2\boldsymbol{a} = \boldsymbol{b}, \quad \text{where} \quad \boldsymbol{a} = \begin{bmatrix} 3, & -1 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 0, & 2 \end{bmatrix}.$$
4. From the equation:
$$3\boldsymbol{c} + 5\boldsymbol{a} = 4\boldsymbol{b}, \quad \text{where} \quad \boldsymbol{a} = \begin{bmatrix} 2, & 4 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 1, & -2 \end{bmatrix}.$$

Question 8. Calculate the side lengths of the triangle $\triangle ABC$. Additionally, find the coordinates of a fourth point D such that the quadrilateral $ABCD$ forms a parallelogram.

1. $A = (-4, -2),\ B = (-1, 4), C = (2, 2)$
2. $A = (5, 1),\ B = (4, 2), C = (-1, 4)$
3. $A = (3, 2),\ B = (7, 4), C = (5, 6)$
4. $A = (-1, 2),\ B = (-3, 4), C = (-3, -3)$

Question 9. Calculate the lengths of the sides and diagonals of the specified quadrilateral $ABCD$, classify its type.

1. $A=(8,-4),\ B=(5,-6),\ C=(1,-4),\ D=(4,2)$
2. $A=(-2,-3),\ B=(-5,-7),\ C=(-1,-10),\ D=(2,-6)$
3. $A=(7,5),\ B=(-6,2),\ C=(3,-1),\ D=(6,1)$

Question 10. Find the interior angles of the triangle $\triangle ABC$, where

$$A=(5\sqrt{3},5),\ B=(-\sqrt{3},1),\ C=(0,0)$$

Question 11. Identify which of the following vector pairs are parallel.

1. $\begin{bmatrix}-2, & 3, & 1\end{bmatrix},\ \begin{bmatrix}6, & -4, & -3\end{bmatrix}$
2. $\begin{bmatrix}10, & -8, & 9, & 0, & 24\end{bmatrix},\begin{bmatrix}\frac{5}{6}, & -\frac{2}{3}, & \frac{3}{4}, & 0, & 2\end{bmatrix}$
3. $\begin{bmatrix}8, & -6, & -9, & -8, & -2\end{bmatrix},\begin{bmatrix}-\frac{32}{3}, & 8, & 12, & \frac{32}{3}, & \frac{8}{3}\end{bmatrix}$
4. $\begin{bmatrix}0, & 0, & 2, & 3, & 5\end{bmatrix},\begin{bmatrix}0, & 1, & \frac{4}{3}, & 2, & \frac{10}{3}\end{bmatrix}$

Question 12. Compute $\text{proj}_{\boldsymbol{a}}\boldsymbol{b}$, $\boldsymbol{b}-\text{proj}_{\boldsymbol{a}}\boldsymbol{b}$, $\text{proj}_{\boldsymbol{b}}\boldsymbol{a}$ and $\boldsymbol{a}-\text{proj}_{\boldsymbol{b}}\boldsymbol{a}$.

1. $\boldsymbol{a}=\begin{bmatrix}2, & 1, & 5\end{bmatrix},\ \boldsymbol{b}=\begin{bmatrix}1, & 4, & -5\end{bmatrix}$
2. $\boldsymbol{a}=\begin{bmatrix}6, & 2, & 4\end{bmatrix},\ \boldsymbol{b}=\begin{bmatrix}1, & -5, & 4\end{bmatrix}$
3. $\boldsymbol{a}=\begin{bmatrix}1, & 0, & 1, & -2\end{bmatrix},\ \boldsymbol{b}=\begin{bmatrix}5, & -1, & 0, & 2\end{bmatrix}$
4. $\boldsymbol{a}=\begin{bmatrix}-2, & 3, & -4, & 1\end{bmatrix},\ \boldsymbol{b}=\begin{bmatrix}5, & 1, & -8, & 2\end{bmatrix}$

Question 13. For any $\boldsymbol{a},\boldsymbol{b}\in\mathbb{R}^n$, show that $\boldsymbol{b}-\text{proj}_{\boldsymbol{a}}\boldsymbol{b}$ is orthogonal to $\boldsymbol{a}$.

Question 14. Take any $\boldsymbol{a},\boldsymbol{b}\in\mathbb{R}^n$, where both vectors are nonzero.

1. Suppose $\boldsymbol{a}$ and $\boldsymbol{b}$ are orthogonal. Find $\text{proj}_{\boldsymbol{a}}\boldsymbol{b}$.
2. Suppose $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel. Find $\text{proj}_{\boldsymbol{a}}\boldsymbol{b}$.

2 Matrices

Question 1. Let

$$A = \begin{bmatrix} -4 & 2 & 3 \\ 0 & 5 & -1 \\ 6 & 1 & -2 \end{bmatrix} \quad B = \begin{bmatrix} 6 & -1 & 0 \\ 2 & 2 & -4 \\ 3 & -1 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 5 & -1 \\ -3 & 4 \end{bmatrix}$$

$$D = \begin{bmatrix} -7 & 1 & -4 \\ 3 & -2 & 8 \end{bmatrix} \quad E = \begin{bmatrix} 3 & -3 & 5 \\ 1 & 0 & -2 \\ 6 & 7 & -2 \end{bmatrix} \quad F = \begin{bmatrix} 8 & -1 \\ 2 & 0 \\ 5 & -3 \end{bmatrix}$$

Compute, if possible, the following matrices.

1. $A + B$
2. $2A - 3E - B$
3. $2C^\top - 3F$
4. $C + D$
5. $2D - 3F$
6. $5(F^\top - D^\top)$
7. $4A$
8. $A^\top + E^\top$
9. $\left((B - A)^\top + E^\top\right)^\top$
10. $2A - 3B$
11. $(A + E)^\top$
12. $A - B + E$
13. $C + 3F - E$
14. $4D + 3F^\top$

For Questions 2-5, let

$$A = \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix}, \quad E = \begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix}$$

Question 2. Compute, if possible, the following matrices.

1. $D + E$
2. $-3(D + 2E)$
3. $D - E$
4. $A - A$
5. $5A$
6. $\mathrm{tr}\,(D)$
7. $-7D$
8. $\mathrm{tr}\,(D - 3E)$
9. $2B - C$
10. $4\mathrm{tr}\,(7B)$
11. $4E - 2D$
12. $\mathrm{tr}\,(A)$

Question 3. Compute, if possible, the following matrices.

1. $2A^\top + C$
2. $2E^\top - 3D^\top$
3. $D^\top - E^\top$
4. $(2E^\top - 3D^\top)^\top$
5. $(D - E)^\top$
6. $(CD)E$
7. $B^\top + 5C^\top$
8. $C(AB)$
9. $\frac{1}{2}C^\top - \frac{1}{4}A$
10. $\operatorname{tr}\left(DE^\top\right)$
11. $B - B^\top$
12. $\operatorname{tr}(BC)$

Question 4. Compute, if possible, the following matrices.

1. AB
2. BA
3. $(3E)D$
4. $(AB)C$
5. $A(BC)$
6. $CC^\top$
7. $(DA)^\top$
8. $(C^\top B)A^\top$
9. $\operatorname{tr}\left(DD^\top\right)$
10. $\operatorname{tr}\left(4E^\top - D\right)$
11. $\operatorname{tr}\left(C^\top A^\top + 2E^\top\right)$
12. $\operatorname{tr}\left((EC^\top)^\top A\right)$

Question 5. Compute, if possible, the following matrices.

1. $(2D^\top - E)A$
2. $(BA^\top - 2C)^\top$
3. $(4B)C + 2B$
4. $B^\top(CC^\top - A^\top A)$
5. $(-AC)^\top + 5D^\top$
6. $D^\top E^\top - (ED)^\top$

Question 6. Given matrices

$$A \in \mathcal{M}_{4\times 5}, \quad B \in \mathcal{M}_{4\times 5}, \quad C \in \mathcal{M}_{5\times 2}, \quad D \in \mathcal{M}_{4\times 2}, \quad E \in \mathcal{M}_{5\times 4},$$

for each of the cases below, determine whether the given computation is defined. If it is, specify the size of the resulting matrix.

1. BA
2. $A - 3E^\top$
3. $BC - 3D$
4. $AB^\top$
5. $E(5B + A)$
6. $D^\top(BE)$
7. $AC + D$
8. $CD^\top$
9. $B^\top D + ED$
10. $E(AC)$
11. DC
12. $BA^\top + D$

Question 7. Determine which of the following matrices are square, diagonal, upper or lower triangular, symmetric, or skew-symmetric. Compute the transpose of each matrix.

$$A = \begin{bmatrix} -1 & 4 \\ 0 & 1 \\ 6 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}, \quad C = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} -1 \\ 4 \\ 2 \end{bmatrix}$$

$$E = \begin{bmatrix} 0 & 0 & 6 \\ 0 & -6 & 0 \\ -6 & 0 & 0 \end{bmatrix} \qquad F = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} \qquad G = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

$$H = \begin{bmatrix} 0 & -1 & 6 & 2 \\ 1 & 0 & -7 & 1 \\ -6 & 7 & 0 & -4 \\ -2 & -1 & 4 & 0 \end{bmatrix} \qquad J = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \qquad K = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -2 & 1 & 5 & 6 \\ -3 & -5 & 1 & 7 \\ -4 & -6 & -7 & 1 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad M = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} \qquad N = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad Q = \begin{bmatrix} -2 & 0 & 0 \\ 4 & 0 & 0 \\ -1 & 2 & 3 \end{bmatrix} \qquad R = \begin{bmatrix} 6 & 2 \\ 3 & -2 \\ -1 & 0 \end{bmatrix}$$

For Questions 8-13, let

$$A = \begin{bmatrix} -2 & 3 \\ 6 & 5 \\ 1 & -4 \end{bmatrix} \quad B = \begin{bmatrix} -5 & 3 & 6 \\ 3 & 8 & 0 \\ -2 & 0 & 4 \end{bmatrix} \quad C = \begin{bmatrix} 11 & -2 \\ -4 & -2 \\ 3 & -1 \end{bmatrix}$$

$$D = \begin{bmatrix} -1 & 4 & 3 & 7 \\ 2 & 1 & 7 & 5 \\ 0 & 5 & 5 & -2 \end{bmatrix} \quad E = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \quad F = \begin{bmatrix} 9 & -3 \\ 5 & -4 \\ 2 & 0 \\ 8 & -3 \end{bmatrix}$$

$$G = \begin{bmatrix} 5 & 1 & 0 \\ 0 & -2 & -1 \\ 1 & 0 & 3 \end{bmatrix} \quad H = \begin{bmatrix} 6 & 3 & 1 \\ 1 & -15 & -5 \\ -2 & -1 & 10 \end{bmatrix} \quad J = \begin{bmatrix} 8 \\ -1 \\ 4 \end{bmatrix}$$

$$K = \begin{bmatrix} 2 & 1 & -5 \\ 0 & 2 & 7 \end{bmatrix} \quad L = \begin{bmatrix} 10 & 9 \\ 8 & 7 \end{bmatrix} \quad M = \begin{bmatrix} 7 & -1 \\ 11 & 3 \end{bmatrix}$$

$$N = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad P = \begin{bmatrix} 3 & -1 \\ 4 & 7 \end{bmatrix} \quad Q = \begin{bmatrix} 1 & 4 & -1 & 6 \\ 8 & 7 & -3 & 3 \end{bmatrix}$$

$$R = \begin{bmatrix} -3, & 6, & -2 \end{bmatrix} \quad S = \begin{bmatrix} 6, & -4, & 3, & 2 \end{bmatrix} \quad T = \begin{bmatrix} 4, & -1, & 7 \end{bmatrix}$$

Question 8. Compute, if possible, the following matrices.

1. AB
2. BA
3. JM
4. DF
5. RJ
6. JR
7. RT
8. SF
9. KN
10. F^2
11. B^2
12. E^3
13. $(TJ)^3$
14. $D(FK)$
15. $(CL)G$

Question 9. Identify which of the following matrix pairs commute.

1. L and M
2. F and Q
3. G and H
4. R and J
5. A and K
6. N and P

Question 10. Find the specified row or column of the matrix product for the given matrices.

1. The second row of the product BG
2. The third column of the product DE
3. The first column of the product SE
4. The third row of the product FQ

Question 11. Verify whether the following computations can be performed. If they are possible, identify which of the specified equalities hold. Provide justification by naming the relation and referencing the mathematical properties that confirm or disprove the validity of the equality.

1. $(RG)H = R(GH)$
2. $LP = PL$
3. $E(FK) = (EF)K$
4. $K(A + C) = KA + KC$
5. $(QF)^\top = F^\top Q^\top$
6. $L(ML) = L^2M$
7. $GC + HC = (G + H)C$
8. $R(J + T^\top) = RJ + RT^\top$
9. $(AK)^\top = A^\top K^\top$
10. $(Q + F^\top)E^\top = QE^\top + (EF)^\top$

Question 12. Given matrices

$$A = \begin{bmatrix} 3 & -2 & 7 \\ 6 & 5 & 4 \\ 0 & 4 & 9 \end{bmatrix}, \quad B = \begin{bmatrix} 6 & -2 & 4 \\ 0 & 1 & 3 \\ 7 & 7 & 5 \end{bmatrix};$$

or

$$A = \begin{bmatrix} 1 & 5 & -2 & 1 \\ 0 & 1 & -2 & 3 \\ 1 & 1 & 2 & 1 \\ 0 & 0 & 2 & -4 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 & 5 & -1 \\ 2 & 1 & 2 & 1 \\ 4 & -2 & -1 & 0 \\ -1 & 2 & 1 & 3 \end{bmatrix}$$

Compute the specified row or column of the matrix product of the given matrices, both for the 3×3 and 4×4 cases:

1. The first row of the product AB
2. The third row of the product AB
3. The second column of the product AB
4. The first column of the product BA
5. The second column of the product BA
6. The third row of the product AA
7. The third column of the product AA

8. The first column of the product AA
9. The second column of the product BB
10. The second row of the product BB
11. The third column of the product BAA

Question 13.

1. Find a non-diagonal matrix A such that $A^2 = I_2$.
2. Find a non-diagonal matrix A such that $A^2 = I_3$ (modify the result from part (a)).
3. Find a non-identity matrix A such that $A^3 = I_3$.