Algebra and Discrete Mathematics
ADM_A_B

doc. Xiaolu Hou, PhD.

FIIT, STU
xiaolu.hou @ stuba.sk

Course Outline

- Vectors and matrices
- System of linear equations
- Matrix inverse and determinants
- Vector spaces, bases and matrix transformations
- Quiz one
- Matrix transformations, fundamental spaces and decompositions
- Eulerian tours and Hamiltonian cycles
- Paths and spanning trees
- Quiz two
- Trees and networks
- Matching

Recommended reading

- Saoub, K. R. (2017). A tour through graph theory. Chapman and Hall/CRC.
  - Sections 3.1, 3.2, 4.1, 4.2
  - Accessible online (free copy)
  - Alternative download link

Lecture outline

- Dijkstra's Algorithm
- Chinese Postman Problem
- Project scheduling
- Critical path
- Trees
- Spanning trees

Paths and spanning trees

- Dijkstra's Algorithm
- Chinese Postman Problem
- Project scheduling
- Critical path
- Trees
- Spanning trees

Shortest path problem

- A path is a sequence of vertices in which there is an edge between consecutive vertices and no vertex is repeated
- Weight: distance, cost, time, etc.
- Shortest path: the path of the least total weight
- A shortest path between any two vertices exists if the graph is connected
- Scenario: fastest route to travel from one location to another

Dijkstra's Algorithm

- Proposed in 1956 by Edsger W. Dijkstra.
- Almost every GIS (Geographic Information System, or mapping software) uses a modification of Dijkstra's algorithm to provide directions
- Numerous versions of Dijkstra's Algorithm exist
- See original algorithm: DIJKSTRA, E. (1959). A Note on Two Problems in Connexion with Graphs. Numerische Mathematik, 1, 269-271.

Dijkstra's Algorithm – notations

- Each vertex is given a two-part label

$$L(v) = (x, \omega(v))$$

- x: the predecessor of v on the current best known path
- $\omega(v)$: the weight of the path that was used to get to v from the designated starting vertex
- F: a set of vertices that are not highlighted yet

Dijkstra's Algorithm – input and output

- **Input**: Weighted connected graph $G = (V, E)$ and vertices designated as *Start* and *End*
- **Output**: Highlighted path from *Start* to *End* and its total weight $\omega(\mathit{End})$

Dijkstra's Algorithm – steps

1. For each vertex v of G, assign a label $L(v)$:

$$L(v) = \begin{cases} (-, 0), & \text{if } v \text{ is } \textit{Start} \\ (-, \infty), & \text{Otherwise} \end{cases}$$

   Highlight *Start*. $F = V - \{\textit{Start}\}$. Let the *current vertex* be *Start*.

2. Update the labels for each vertex v in F that is a neighbor of *current vertex*, say u:

$$L(v) = \begin{cases} (u, \omega(u) + \omega(uv)), & \text{if } \omega(u) + \omega(uv) < \omega(v) \\ L(v), & \text{Otherwise} \end{cases}$$

3. Highlight the vertex v in F with the lowest weight as well as the edge used to update the label. Remove v from F. Redefine the *current vertex* to be v.

4. Repeat steps 2 and 3 until the vertex *End* has been highlighted.

Dijkstra's Algorithm – steps

5. The shortest path from *Start* to *End* is found by tracing back from *End* using the first component of the labels. The total weight of the path is the weight for *End* given in the second component of its label.

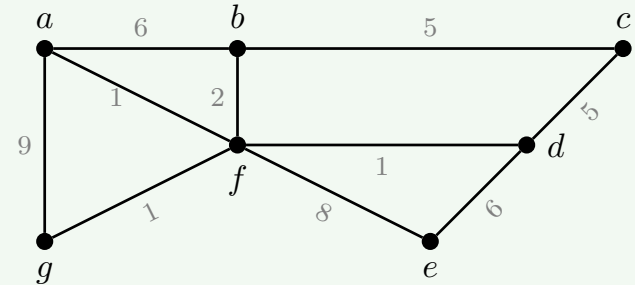
Note

- The set F of vertices consists of all un-highlighted vertices and all are under consideration for becoming the next highlighted vertex
- It is important that we do not only consider the neighbors of the last vertex highlighted, as a path from a previously chosen vertex may in fact lead to the shortest path

Dijkstra's Algorithm – example

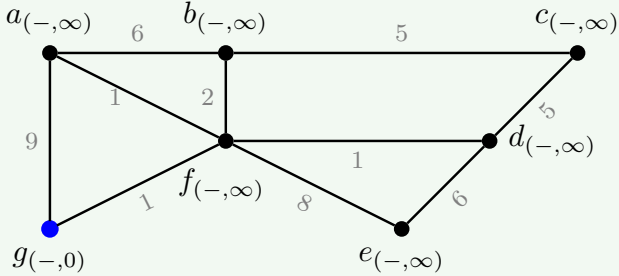
Example

Take *Start* $= g$ and *End* $= c$.



Dijkstra's Algorithm – example

Example



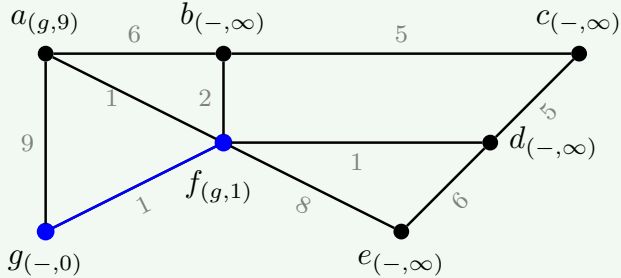
- Step 1. Highlight g. Label

$$L(v) = \begin{cases} (-,0) & v = g \\ (-,\infty) & \text{Otherwise} \end{cases}$$

$F = \{a, b, c, d, e, f\}$. The *current vertex* is g.

Dijkstra's Algorithm – example

Example



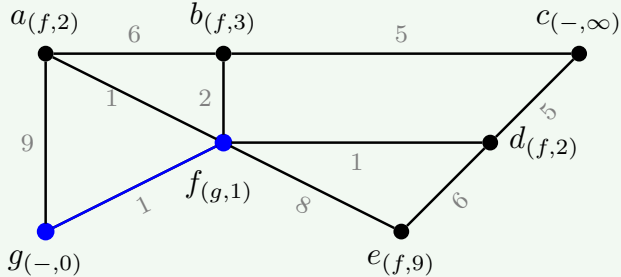
- Step 2. $F = \{a, b, c, d, e, f\}$. The neighbors of g are a and f.

$$\begin{aligned} \omega(g) + \omega(ga) &= 0 + 9 = 9 < \infty = \omega(a) \Longrightarrow L(a) = (g, 9) \\ \omega(g) + \omega(gf) &= 0 + 1 = 1 < \infty = \omega(f) \Longrightarrow L(f) = (g, 1) \end{aligned}$$

- Step 3. minimum weight: f; highlight gf and f. $F = \{a, b, c, d, e\}$. *current vertex* is f.

Dijkstra's Algorithm – example

Example

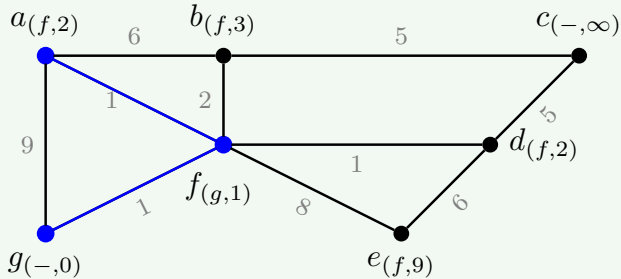


- Step 2. $F = \{a, b, c, d, e\}$. Neighbors of f: a, b, d, e

$$
\begin{aligned}
\omega(f) + \omega(fa) &= 1 + 1 = 2 < 9 = \omega(a) \Longrightarrow L(a) = (f, 2) \\
\omega(f) + \omega(fb) &= 1 + 2 = 3 < \infty = \omega(b) \Longrightarrow L(b) = (f, 3) \\
\omega(f) + \omega(fd) &= 1 + 1 = 2 < \infty = \omega(d) \Longrightarrow L(d) = (f, 2) \\
\omega(f) + \omega(fe) &= 1 + 8 = 9 < \infty = \omega(e) \Longrightarrow L(e) = (f, 9)
\end{aligned}
$$

Dijkstra's Algorithm – example

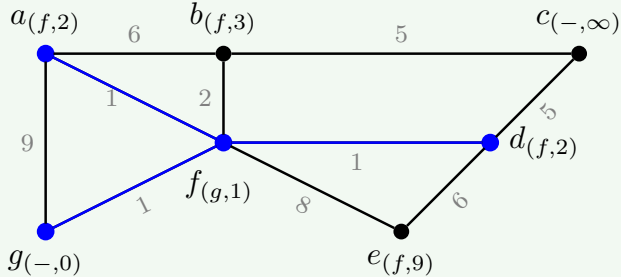
Example



- Step 3. $F = \{a, b, c, d, e\}$. The minimum weight among the vertices in F is attained by both a and d. Choose one of them arbitrarily.
  - Let us highlight fa and a.
  - $F = \{b, c, d, e\}$.
  - *current vertex* is a.

Dijkstra's Algorithm – example

Example



- Step 2. $F = \{b, c, d, e\}$. Neighbor of a: b

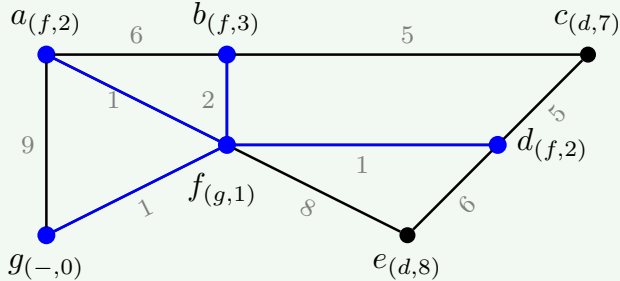
$$\omega(a) + \omega(ba) \quad = \quad 2 + 6 = 8 > 3 = \omega(b)$$

  We do not update the label of b.

- Step 3. Highlight fd and d. $F = \{b, c, e\}$. *current vertex* is d.

Dijkstra's Algorithm – example

Example



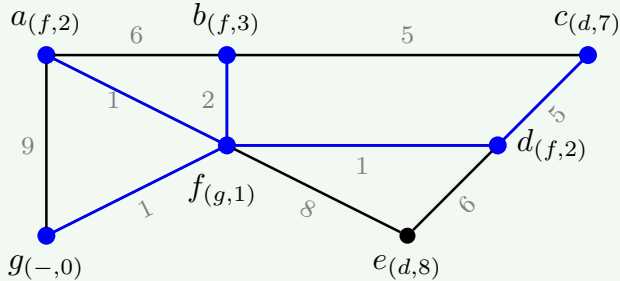
- Step 2. $F = \{b, c, e\}$. Neighbors of d: c, e

$$\begin{aligned} \omega(d) + \omega(dc) &= 2 + 5 = 7 < \infty = \omega(c) \Longrightarrow L(c) = (d, 7) \\ \omega(d) + \omega(de) &= 2 + 6 = 8 < 9 = \omega(e) \Longrightarrow L(e) = (d, 8) \end{aligned}$$

- Step 3. Highlight fb and b. $F = \{c, e\}$. *current vertex* is b.

Dijkstra's Algorithm – example

Example



- Step 2. $F = \{c, e\}$. Neighbor of b: $c\ \omega(b) + \omega(bc) = 3 + 5 = 8 > 7 = \omega(c)$
- Step 3. Highlight dc and c.
- Step 4. This terminates the iterations since we have reached *End*
- Step 5. The shortest path from g to c is $g \to f \to d \to c$, with total weight 7.

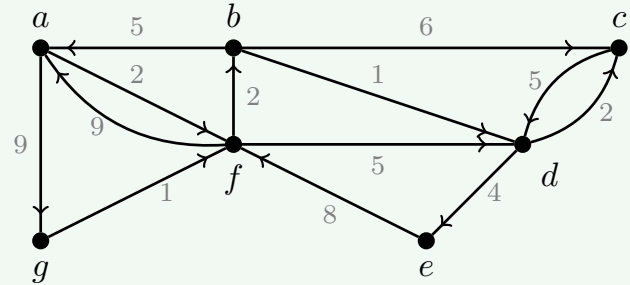
Dijkstra's Algorithm for digraphs

- A digraph is a graph in which the edges now have a direction associated to them, which could be used to model a one-way street.
- For an arc yx, x is the *head* and y is the *tail*.
- In Step 2, instead of all neighbors, we consider only the *out-neighbors* of the current vertex, that is, vertices joined to it by outgoing arcs.

Dijkstra's Algorithm for digraphs – example

Example

Start= g, *End*= c



We record the changes in the following table.

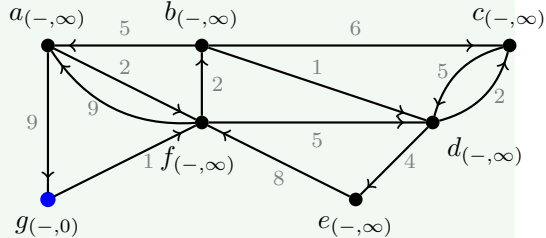
g	a	b	c	d	e	f

Dijkstra's Algorithm for digraphs – example

Example

- Step 1. *current vertex* is g,
  $F = \{a, b, c, d, e, f\}$

g	a	b	c	d	e	f
0	∞	∞	∞	∞	∞	∞



Dijkstra's Algorithm for digraphs – example

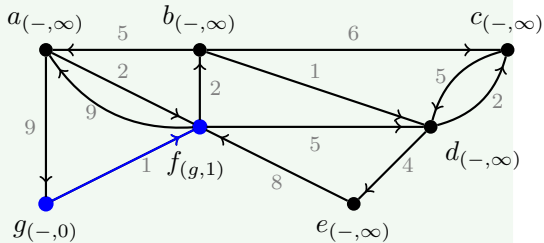
Example

- Step 2. *current vertex* is g,
  $F = \{a, b, c, d, e, f\}$,
  out-neighbors of g: f

$$\omega(g) + \omega(gf) = 0 + 1 < \infty$$

- Step 3. *current vertex* is f,
  $F = \{a, b, c, d, e\}$

	a	b	c	d	e	f
	∞	∞	∞	∞	∞	∞
$g(0)$	∞	∞	∞	∞	∞	$(g, 1)$

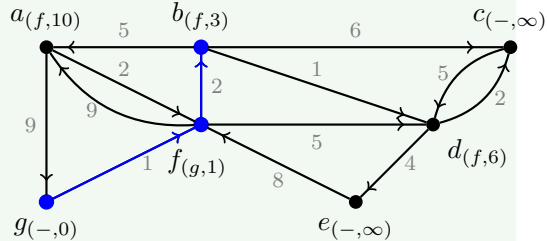


Dijkstra's Algorithm for digraphs – example

Example

- Step 2. *current vertex* is f,
  $F = \{a, b, c, d, e\}$,
  out-neighbors of f: a, b, d

$$\omega(f) + \omega(fa) = 1 + 9 = 10 < \infty,$$
$$\omega(f) + \omega(fb) = 1 + 2 = 3 < \infty,$$
$$\omega(f) + \omega(fd) = 1 + 5 = 6 < \infty.$$



- Step 3. *current vertex* is b,
  $F = \{a, c, d, e\}$

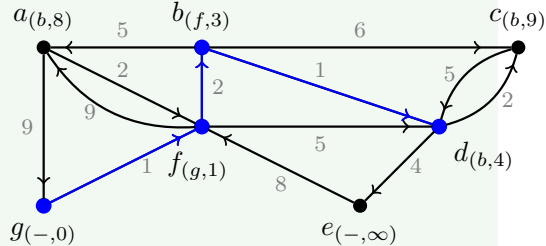
	a	b	c	d	e	f
	∞	∞	∞	∞	∞	∞
$g(0)$	∞	∞	∞	∞	∞	$(g,1)$
$f(1)$	$(f,10)$	$(f,3)$	∞	$(f,6)$	∞	$(g,1)$

Dijkstra's Algorithm for digraphs – example

Example

- Step 2. *current vertex* is b,
  $F = \{a, c, d, e\}$,
  out-neighbors of b: a, c, d

$$\omega(b) + \omega(ba) = 3 + 5 = 8 < 10,$$
$$\omega(b) + \omega(bc) = 3 + 6 = 9 < \infty,$$
$$\omega(b) + \omega(bd) = 3 + 1 = 4 < 6.$$



- Step 3. *current vertex* is d, $F = \{a, c, e\}$

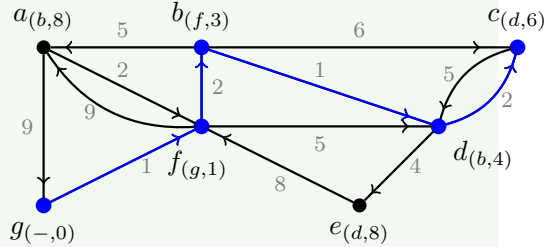
	a	b	c	d	e	f
	∞	∞	∞	∞	∞	∞
$g(0)$	∞	∞	∞	∞	∞	$(g,1)$
$f(1)$	$(f,10)$	$(f,3)$	∞	$(f,6)$	∞	$(g,1)$
$b(3)$	$(b,8)$	$(f,3)$	$(b,9)$	$(b,4)$	∞	$(g,1)$

Dijkstra's Algorithm for digraphs – example

Example

- Step 2. *current vertex* is d, $F = \{a, c, e\}$, out-neighbors of d: c, e

$$\omega(d) + \omega(dc) = 4 + 2 = 6 < 9,$$
$$\omega(d) + \omega(de) = 4 + 4 = 8 < \infty$$



- Step 3. *current vertex* is c
- Step 4. we have reached *End*
- Step 5. $g \to f \to b \to d \to c$, weight 6

	a	b	c	d	e	f
	∞	∞	∞	∞	∞	∞
$g(0)$	∞	∞	∞	∞	∞	$(g,1)$
$f(1)$	$(f,10)$	$(f,3)$	∞	$(f,6)$	∞	$(g,1)$
$b(3)$	$(b,8)$	$(f,3)$	$(b,9)$	$(b,4)$	∞	$(g,1)$
$d(4)$	$(b,8)$	$(f,3)$	$(d,6)$	$(b,4)$	$(d,8)$	$(g,1)$

Remark

- It is possible for a path not to exist from one vertex to another based upon the direction of the arcs
- For example, if a is the head of every arc incident to a, then no directed path can start at a.
- In such a case Dijkstra's Algorithm would halt and note that a shortest path could not be found

Paths and spanning trees

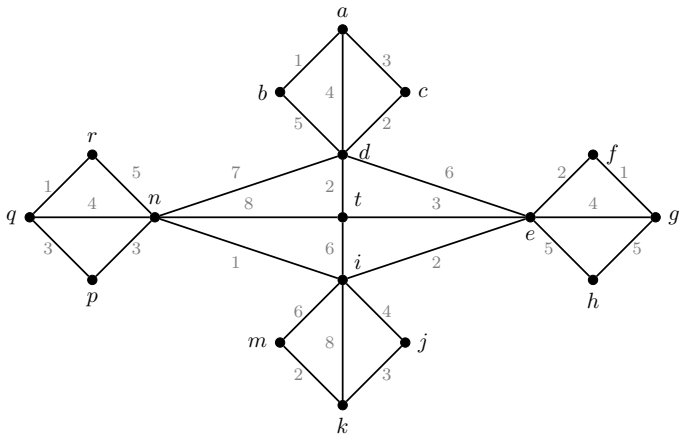
- Dijkstra's Algorithm
- Chinese Postman Problem
- Project scheduling
- Critical path
- Trees
- Spanning trees

Chinese Postman Problem

- Find an Eulerian circuit of minimal total weight
- Eulerization: identify the odd-degree vertices and pair them so as to minimize the total weight of the duplicated paths.
- We now combine Eulerization with Dijkstra's Algorithm to obtain a more systematic solution to the Chinese Postman Problem.

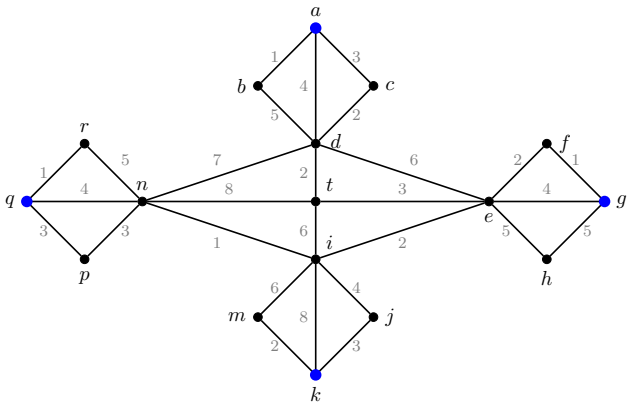
Chinese Postman Problem – example

Use Dijkstra's Algorithm to determine the best pairing of the odd vertices and the total weight of the duplicated edges in an Eulerization of the graph.



Chinese Postman Problem – example

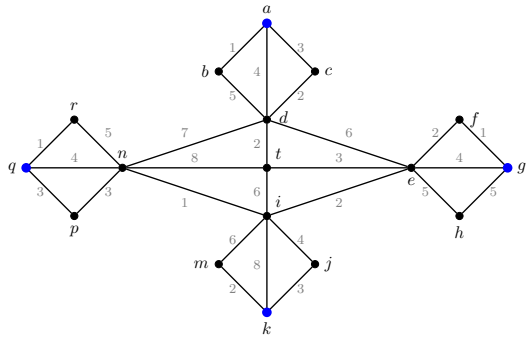
- Four odd vertices: a, g, k, q
- Three possible pairings $a-g, k-q$, or $a-k, g-q$, or $a-q, g-k$



Chinese Postman Problem – example

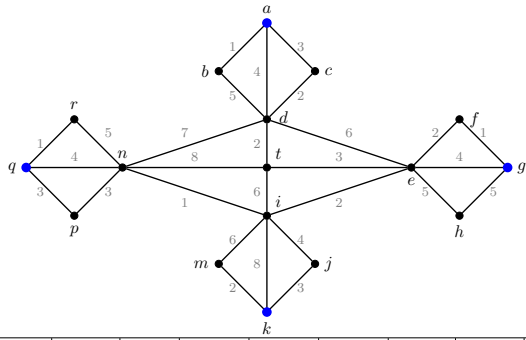
- Three possible pairings $a-g, k-q$, or $a-k, g-q$, or $a-q, g-k$
- Applying Dijkstra's Algorithm, we find the shortest path for each paired set of vertices, and hence the total weight of the duplicated paths needed to Eulerize the graph.

Chinese Postman Problem – example



	b	c	d	n	e	t	r	q	p	f	g	h	i	m	j	k
	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞
$a(0)$	$(a,1)$	$(a,3)$	$(a,4)$	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞
$b(1)$	$(a,1)$	$(a,3)$	$(a,4)$	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞
$c(3)$	$(a,1)$	$(a,3)$	$(a,4)$	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞

Chinese Postman Problem – example



	b	c	d	n	e	t	r	q	p	f	g	h	i	m	j	k
	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞
$a(0)$	$(a,1)$	$(a,3)$	$(a,4)$	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞
$b(1)$	$(a,1)$	$(a,3)$	$(a,4)$	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞
$c(3)$	$(a,1)$	$(a,3)$	$(a,4)$	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞
$d(4)$	$(a,1)$	$(a,3)$	$(a,4)$	$(d,11)$	$(d,10)$	$(d,6)$	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞
$t(6)$	$(a,1)$	$(a,3)$	$(a,4)$	$(d,11)$	$(t,9)$	$(d,6)$	∞	∞	∞	∞	∞	∞	$(t,12)$	∞	∞	∞
$e(9)$	$(a,1)$	$(a,3)$	$(a,4)$	$(d,11)$	$(t,9)$	$(d,6)$	∞	∞	∞	$(e,11)$	$(e,13)$	$(e,14)$	$(e,11)$	∞	∞	∞
$f(11)$	$(a,1)$	$(a,3)$	$(a,4)$	$(d,11)$	$(t,9)$	$(d,6)$	∞	∞	∞	$(e,11)$	$(f,12)$	$(e,14)$	$(e,11)$	∞	∞	∞
$i(11)$	$(a,1)$	$(a,3)$	$(a,4)$	$(d,11)$	$(t,9)$	$(d,6)$	∞	∞	∞	$(e,11)$	$(f,12)$	$(e,14)$	$(e,11)$	$(i,17)$	$(i,15)$	$(i,19)$
$n(11)$	$(a,1)$	$(a,3)$	$(a,4)$	$(d,11)$	$(t,9)$	$(d,6)$	$(n,16)$	$(n,15)$	$(n,14)$	$(e,11)$	$(f,12)$	$(e,14)$	$(e,11)$	$(i,17)$	$(i,15)$	$(i,19)$

Chinese Postman Problem – example

	b	c	d	n	e	t	r	q	p	f	g	h	i	m	j	k
	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞
$a(0)$	$(a,1)$	$(a,3)$	$(a,4)$	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞
$b(1)$	$(a,1)$	$(a,3)$	$(a,4)$	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞
$c(3)$	$(a,1)$	$(a,3)$	$(a,4)$	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞
$d(4)$	$(a,1)$	$(a,3)$	$(a,4)$	$(d,11)$	$(d,10)$	$(d,6)$	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞
$t(6)$	$(a,1)$	$(a,3)$	$(a,4)$	$(d,11)$	$(t,9)$	$(d,6)$	∞	∞	∞	∞	∞	∞	$(t,12)$	∞	∞	∞
$e(9)$	$(a,1)$	$(a,3)$	$(a,4)$	$(d,11)$	$(t,9)$	$(d,6)$	∞	∞	∞	$(e,11)$	$(e,13)$	$(e,14)$	$(e,11)$	∞	∞	∞
$f(11)$	$(a,1)$	$(a,3)$	$(a,4)$	$(d,11)$	$(t,9)$	$(d,6)$	∞	∞	∞	$(e,11)$	$(f,12)$	$(e,14)$	$(e,11)$	∞	∞	∞
$i(11)$	$(a,1)$	$(a,3)$	$(a,4)$	$(d,11)$	$(t,9)$	$(d,6)$	∞	∞	∞	$(e,11)$	$(f,12)$	$(e,14)$	$(e,11)$	$(i,17)$	$(i,15)$	$(i,19)$
$n(11)$	$(a,1)$	$(a,3)$	$(a,4)$	$(d,11)$	$(t,9)$	$(d,6)$	$(n,16)$	$(n,15)$	$(n,14)$	$(e,11)$	$(f,12)$	$(e,14)$	$(e,11)$	$(i,17)$	$(i,15)$	$(i,19)$

$$a \to d \to t \to e \to f \to g$$

with total weight 12

Chinese Postman Problem – example

- Three possible pairings $a-g, k-q$, or $a-k, g-q$, or $a-q, g-k$
- Applying Dijkstra's Algorithm, we find the shortest path for each paired set of vertices, and hence the total weight of the duplicated paths needed to Eulerize the graph.

Path Pairs	**Weight**	**Total Weight**
$a \to d \to t \to e \to f \to g$	12	
$k \to j \to i \to n \to q$	12	24
$a \to d \to t \to e \to i \to j \to k$	18	
$g \to f \to e \to i \to n \to q$	10	28
$a \to d \to n \to q$	15	
$g \to f \to e \to i \to j \to k$	12	27

Exercise: verify these shortest paths and total weights.

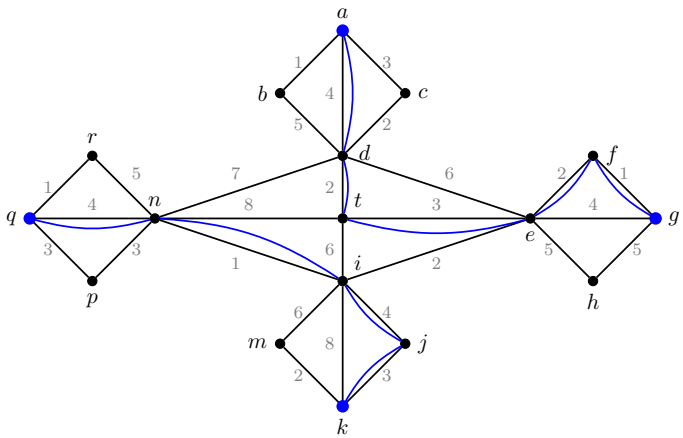
Chinese Postman Problem – example

- Once the shortest paths have been found, we duplicate the edges along those paths to obtain an Eulerization. the Eulerization
- In doing so, we must check whether the two paths in a pairing share any edges. in both paths of a pairing
- For example, the paths in the second pairing both use the edge ei
- We should never duplicate an edge more than once during an Eulerization
- We therefore modify the chosen pair of paths so that the edge ei is duplicated only once.
- This preserves the degree condition, so that all vertices remain even, and reduces the total added weight by 4, making this Eulerization equal in cost to the first one.

$adteijk$	18	
$gfeinq$	10	28

Chinese Postman Problem – example

- $a \to d \to t \to e \to f \to g$, $k \to j \to i \to n \to q$
- $a \to d \to t \to e \to i \to j \to k$, $g \to f \to e \to i \to n \to q$



Remarks

- The use of Dijkstra's Algorithm allows for a methodical approach to the Chinese Postman Problem.
- However, even for small examples, the number of shortest paths that must be considered can be quite large.
- A more efficient approach uses matchings.

Paths and spanning trees

- Dijkstra's Algorithm
- Chinese Postman Problem
- Project scheduling
- Critical path
- Trees
- Spanning trees

Definitions

Definition

Consider a project consisting of multiple tasks.

- *Task*: a required step of a project that cannot be broken into smaller pieces. Labeled with lowercase letters
- *Processor*: the unit (such as a person) that completes a task. Labeled as P_1, P_2, etc. At any time, a processor is either idle or busy performing a task.
- At any stage of a project, a task can be in one of four states:
  - *eligible*: the task can be performed
  - *ineligible*: the task cannot be performed
  - *in execution*: the task is currently being performed
  - *completed*: the task has been completed

  A task is eligible when all tasks on which it depends have been completed.

Definitions

Definition

Consider a project containing multiple parts of steps.

- *Processing time* of a task: the time it takes to complete the task, denoted by $\mathrm{pt}(v)$ for task v
- *Precedence relationship*: task b relies on the completion of task a before it can be eligible
- *Finishing time* of a schedule is the total time used in that schedule
- *Optimal time* of a project is the minimum finishing time among all possible schedules, denoted OPT.

Tasks and schedules – example

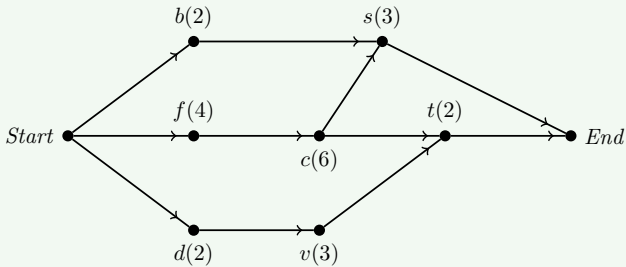
Example

- Party planning

Task	Vertex Name	Processing Time	Precedence Relationships
Buy Food	f	40	
Buy Drinks	b	20	
Dust	d	20	
Vacuum	v	30	d
Cook Food	c	60	f
Set Out Drinks	s	30	b, c
Set Table	t	20	v, c

Tasks and schedules – example

Example



- It is customary to include vertices representing the start and end of the project, and to place vertices so as to avoid edge crossings whenever possible.
- The processing times are shown in parentheses next to the vertex labels
- Edges: precedence relationships

Priority List Model

- Once a digraph has been created, the next step is to determine which processor (or person) should complete each task
- This may be easy for a project with only a few tasks, or when the dependencies between tasks are not complex.
- As complexity grows, we will need a procedure for assigning tasks
- *Priority List Model*:
  - tasks must be assigned to processors according to their order in the *priority list*
  - precedence relationships (displayed in the digraph) are used to determine eligibility

Priority List Model – example

Example

- Continuing from the previous example
- Take priority list: $b - d - t - v - s - f - c$
- Consider two processors
- Each step represents a moment in time where a decision must be made

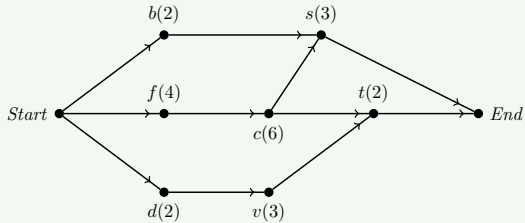
	10	20	30	40	50	60	70	80	90	100	110	120	130	140	150
P_1	b	b													
P_2	d	d													

- Step 1. $T = 0$ The first item in the priority list is b, since b does not rely on any other task, assign it to P_1. The next item d is also eligible. Assign d to P_2.

Priority List Model – example

Example

Priority list: $b - d - t - v - s - f - c$



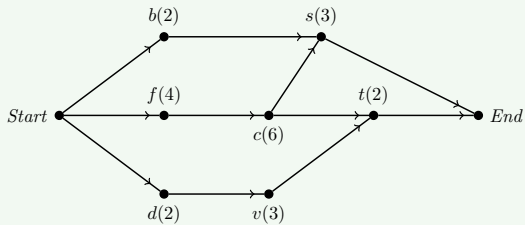
	10	20	30	40	50	60	70	80	90	100	110	120	130	140	150
P_1	b	b	v	v	v										
P_2	d	d	f	f	f	f									

- Step 2. $T = 20$. The next point at which a processor is free to pick up a task is at 20 minutes. The next task on the list is t, but it is ineligible. Both v and f are eligible.

Priority List Model – example

Example

Priority list: $b - d - t - v - s - f - c$



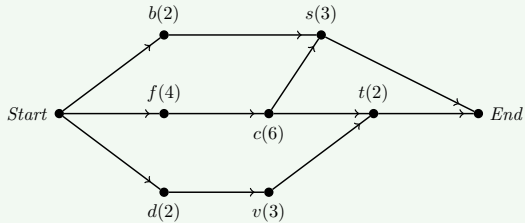
	10	20	30	40	50	60	70	80	90	100	110	120	130	140	150
P_1	b	b	v	v	v	$*$									
P_2	d	d	f	f	f	f									

- Step 3. $T = 50$. At 50 minutes, Processor 1 is ready for a new task. All tasks remaining require f to be complete. P_1 will remain idle until f is complete

Priority List Model – example

Example

Priority list: $b - d - t - v - s - f - c$



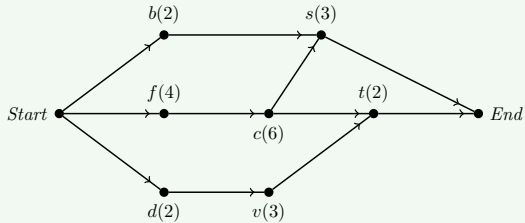
	10	20	30	40	50	60	70	80	90	100	110	120	130	140	150
P_1	b	b	v	v	v	$*$	c	c	c	c	c	c			
P_2	d	d	f	f	f	f	$*$	$*$	$*$	$*$	$*$	$*$			

- Step 4. $T = 60$. At 60 minutes, both processors are ready to take up a new task. The only eligible task is c. By convention, we assign the task to the lower indexed processor. The other processor remains idle

Priority List Model – example

Example

Priority list: $b - d - t - v - s - f - c$



	10	20	30	40	50	60	70	80	90	100	110	120	130	140	150
P_1	b	b	v	v	v	$*$	c	c	c	c	c	c	t	t	$*$
P_2	d	d	f	f	f	f	$*$	$*$	$*$	$*$	$*$	$*$	s	s	s

- Step 5. $T = 120$. Once c is complete, we can assign t to P_1 and s to P_2

The priority list $b - d - t - v - s - f - c$ yields a finishing time of 150 minutes using two processors.

Remarks

- The schedule we obtained contains a large amount of idle time, 80 minutes in total
- Although some idle time may be unavoidable, its presence should indicate that more investigation is warranted.
- The priority list given did not seem to have any connection with the digraph (in fact, it was generated randomly)
- A better approach would be to use information from the digraph to obtain a good priority list.

Paths and spanning trees

- Dijkstra's Algorithm
- Chinese Postman Problem
- Project scheduling
- Critical path
- Trees
- Spanning trees

Critical path

- *Critical path*: a path from *Start* to *End* with the greatest total processing time.
- This path is of interest because it easily identifies restrictions on the completion time of a project
- In addition, it indicates which tasks should be prioritized.
- To find the critical path, we first need to find the *critical times* of all vertices in the graph.

Critical time

Definition

The *critical time* $\mathrm{ct}[x]$ of a vertex x is the processing time of x plus the maximum critical time among all out-neighbors y of x.

$$\mathrm{ct}[x] = \mathrm{pt}(x) + \max\{\, \mathrm{ct}[y] \;\mid\; xy \text{ is an arc} \,\}$$

Note

In the definition, y is an out-neighbor of x

Critical Path Algorithm

Input: Project digraph G with processing times given
Steps:

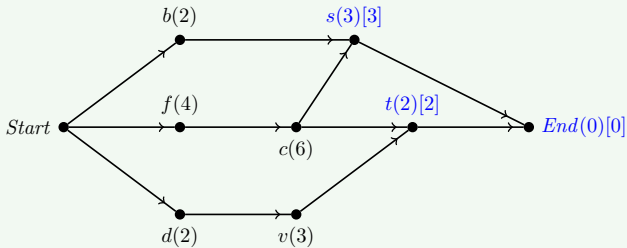
1. Label the vertex *End* with $\mathrm{pt}(\textit{End}) = 0$ and $\mathrm{ct}[\textit{End}] = 0$. For any vertex x with an arc to *End*, define $\mathrm{ct}[x] = \mathrm{pt}(x)$.
2. Travel the arcs in reverse order. When a new vertex is encountered, calculate its critical time.
3. Once all critical times have been obtained, find a path from *Start* to *End* by always following, from the current vertex, an outgoing arc to a neighbor of largest critical time.
4. Create a priority list by ordering vertices by decreasing critical time

Output: Critical path and critical path priority list

Critical Path Algorithm – example

Example

- Continuing from the previous example
- We will use brackets for the critical times, distinguishing them from the processing times

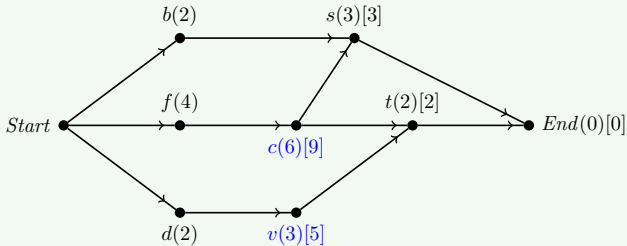


- Step 1. Label *End* with critical times 0. Since s and t have arcs to *End*, set

$$\mathrm{ct}[s] = \mathrm{pt}(s) = 3, \quad \mathrm{ct}[t] = \mathrm{pt}(t) = 2$$

Critical Path Algorithm – example

Example



- Step 2. As v has a single arc to t

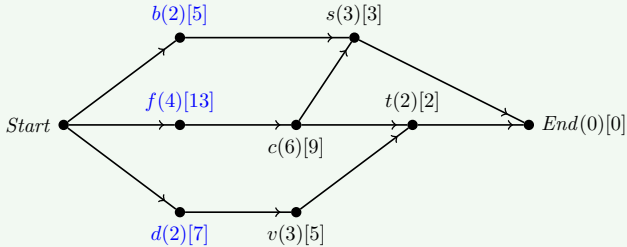
$$\mathrm{ct}[v] = \mathrm{pt}(v) + \mathrm{ct}[t] = 3 + 2 = 5$$

c has an arc to both s and t. $\mathrm{ct}[s] > \mathrm{ct}[t]$

$$\mathrm{ct}[c] = \mathrm{pt}(c) + \mathrm{ct}[s] = 6 + 3 = 9$$

Critical Path Algorithm – example

Example

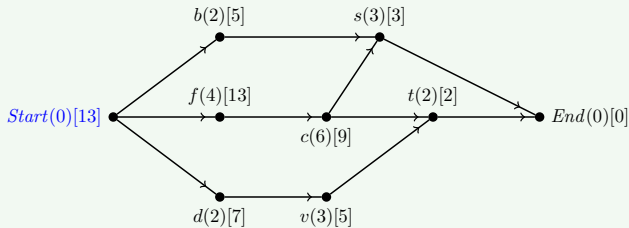


- Step 3. The remaining vertices each have a single arc to previously considered vertices

$$\mathrm{ct}[b] = \mathrm{pt}(b) + \mathrm{ct}[s] = 2 + 3 = 5$$
$$\mathrm{ct}[f] = \mathrm{pt}(f) + \mathrm{ct}[c] = 4 + 9 = 13$$
$$\mathrm{ct}[d] = \mathrm{pt}(d) + \mathrm{ct}[v] = 2 + 5 = 7$$

Critical Path Algorithm – example

Example

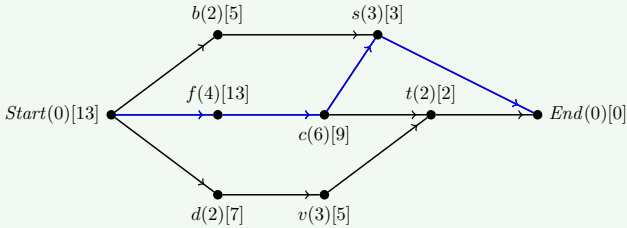


- Step 4. Label the processing time of *Start* as 0. f is the out-neighbor with the largest critical time

$$\mathrm{ct}[\mathit{Start}] = 0 + 13 = 13$$

Critical Path Algorithm – example

Example



- Step 5. Follow the path from *Start* to *End* where the vertices are chosen based on the largest critical times. This gives the path

$$Start \to f \to c \to s \to End$$

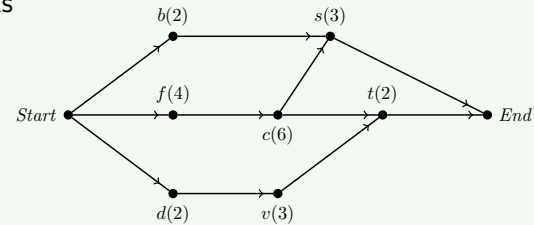
with total time 130. Ordering the vertices by decreasing critical time gives the critical path priority list

$$f - c - d - b - v - s - t$$

Priority List Model for project scheduling – example

Example

Now we use the critical path priority list $f - c - d - b - v - s - t$ to find a new schedule for the tasks



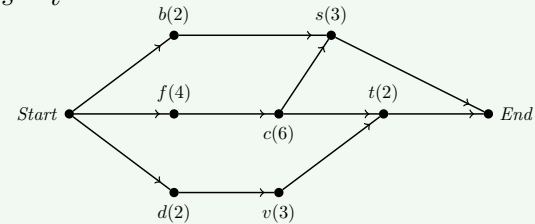
	10	20	30	40	50	60	70	80	90	100	110	120	130
P_1	f	f	f	f									
P_2	d	d											

- Step 1. $T = 0$. Since f is the first item in the list, assign it to P_1. c is not eligible. Assign d to P_2

Priority List Model for project scheduling – example

Example

$f - c - d - b - v - s - t$



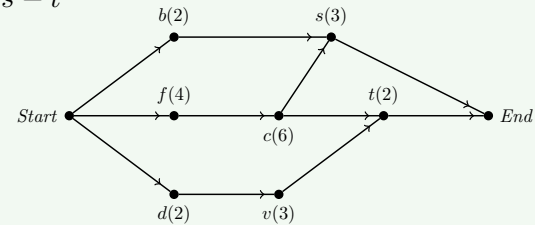
	10	20	30	40	50	60	70	80	90	100	110	120	130
P_1	f	f	f	f									
P_2	d	d	b	b									

- Step 2. $T = 20$. P_2 can be assigned a new task. Assign b, the next eligible task to P_2

Priority List Model for project scheduling – example

Example

$f - c - d - b - v - s - t$



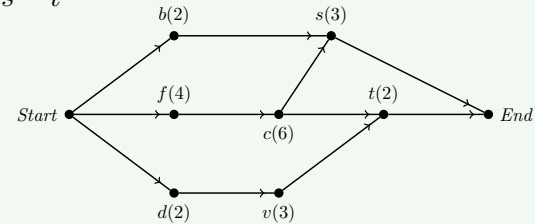
	10	20	30	40	50	60	70	80	90	100	110	120	130
P_1	f	f	f	f	c	c	c	c	c	c			
P_2	d	d	b	b	v	v	v						

- Step 3. $T = 40$. c is eligible. The next eligible task is v

Priority List Model for project scheduling – example

Example

$f - c - d - b - v - s - t$



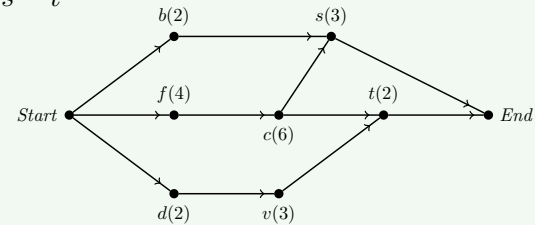
	10	20	30	40	50	60	70	80	90	100	110	120	130
P_1	f	f	f	f	c	c	c	c	c	c			
P_2	d	d	b	b	v	v	v	$*$	$*$	$*$			

- Step 4. $T = 70$. All remaining tasks are ineligible since they rely on the completion of c. P_2 remains idle.

Priority List Model for project scheduling – example

Example

$f - c - d - b - v - s - t$



	10	20	30	40	50	60	70	80	90	100	110	120	130
P_1	f	f	f	f	c	c	c	c	c	c	s	s	s
P_2	d	d	b	b	v	v	v	$*$	$*$	$*$	t	t	$*$

- Step 5. $T = 100$. s and t are now eligible
- Finishing time: 130 minutes, 40 minutes of idle time

Remarks

- Both schedules contained some idle time, though the one using the critical path priority list had only half as much as the initial example. This is partly because tasks on the critical path were prioritized.
- The schedule above must be optimal since its finishing time is equal to the critical time of Start
- In general, the critical path priority list results in a very good, though not always optimal, schedule

Optimal schedule

- The optimal time of a schedule is no less than the critical time of *Start*

$$OPT \geq \mathrm{ct}[\mathit{Start}]$$

- Calculate the sum of all processing times of all tasks. The optimal time is no less than this sum divided by the total number of processors used

$$OPT \geq \frac{\sum_v \mathrm{pt}(v)}{\text{number of processors}}$$

Optimal schedule – example

Example

With our running example, sum of all processing times is 220 minutes

- Using two processors $OPT \geq \frac{220}{2} = 110$
- Using three processors $OPT \geq \frac{220}{3} \approx 73$
- $\mathrm{ct}[\mathit{Start}] = 130$. Neither of the above two calculations provides additional insight into the optimal schedule.

Remark

The calculations also show that two processors are sufficient; additional processors are not necessary.

Paths and spanning trees

- Dijkstra's Algorithm
- Chinese Postman Problem
- Project scheduling
- Critical path
- Trees
- Spanning trees

Definition

Definition

A graph G is

- *acyclic* if there are no cycles or circuits in the graph
- a *network* if it is connected
- a *tree* if it is an acyclic network, i.e. acyclic and connected
- a *forest* if it is an acyclic graph

A vertex of degree 1 is called a *leaf*.

Trees – example

Example

Game

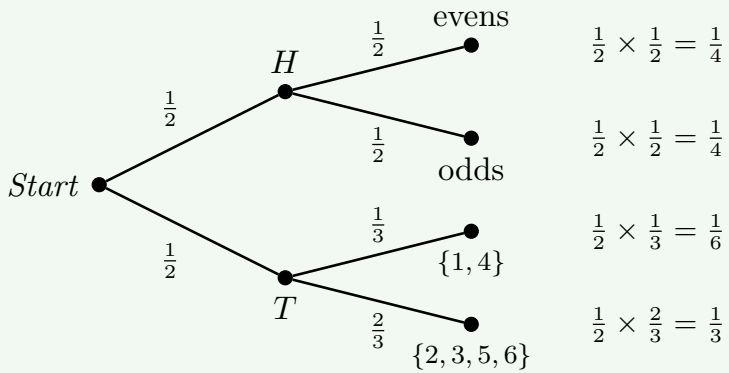
- Adam flips a coin, Jano rolls a die
- Adam gets heads, Jano rolls an even number $\rightarrow$ Jano wins 2 Euros
- Adam gets heads, Jano rolls an odd number $\rightarrow$ Adam wins 3 Euros
- Adam gets tails, Jano rolls 1 or 4 $\rightarrow$ Jano wins 5 Euros
- Adam gets tails, Jano rolls $2, 3, 5$, or 6 $\rightarrow$ Adam wins 2 Euros

A probability tree – vertices representing possible outcomes, edges labeled with the probability of the outcome

Trees – example

Example

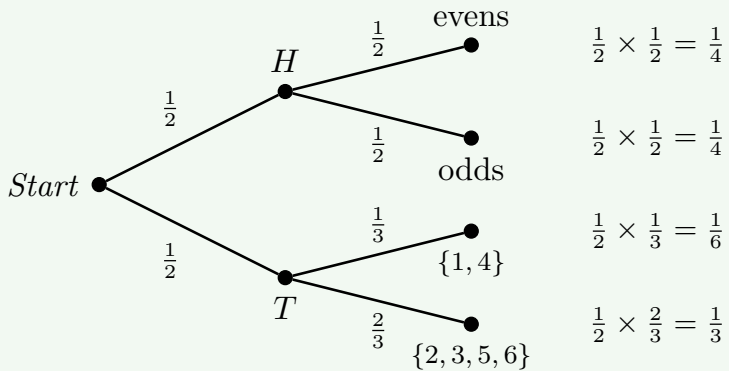
- Adam gets heads, Jano rolls an even number $\rightarrow$ Jano wins 2 Euros
- Adam gets heads, Jano rolls an odd number $\rightarrow$ Adam wins 3 Euros
- Adam gets tails, Jano rolls 1 or 4 $\rightarrow$ Jano wins 5 Euros
- Adam gets heads, Jano rolls $2, 3, 5$, or 6 $\rightarrow$ Adam wins 2 Euros



Trees – example

Example

- The probability Jano wins 5 Euros (tails and 1 or 4) is $\frac{1}{6}$
- The probability that Jano wins any money is $\frac{1}{6} + \frac{1}{4} = \frac{5}{12}$



Trees – example

Example

- Trees can be used to store information for quick access
- Consider the following sequence of numbers

$$4,\ 2,\ 7,\ 10,\ 1,\ 3, 5$$

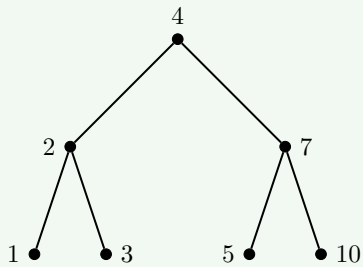
- We can form a tree by creating a vertex for each number in the list
- As we move from one entry in the list to the next, we place an item below and to the left if it is less than the previously viewed vertex and below and to the right if it is greater
- If we add the restriction that no vertex can have more than two edges coming down from it, then we are forming a binary tree.

Trees – example

Example

- As we move from one entry in the list to the next, we place an item below and to the left if it is less than the previously viewed vertex and below and to the right if it is greater
- Restriction: no vertex can have more than two edges coming down from it

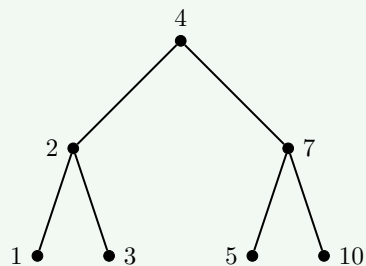
$$4,\ 2,\ 7,\ 10,\ 1,\ 3,\ 5$$



Trees – example

Example

- If we want to search for an item, we may need far fewer comparisons than in a linear search.
- Find item 5, first compare it to the vertex at the top of the tree $\rightarrow$ move along the edge to the right of 4 $\rightarrow$ compare to 7 $\rightarrow$ move along the edge to the left of 7
- Only two comparisons



Properties of trees

1. For every $n \geq 1$, any tree with n vertices has $n - 1$ edges
2. For any tree with $n \geq 1$ vertices, the sum of the degrees is $2n - 2$
3. Every tree with at least two vertices contains at least two leaves
4. Any network on n vertices with $n - 1$ edges must be a tree
5. For any two vertices in a tree, there is a unique path between them
6. The removal of any edge of a tree will disconnect the graph

Note

- $1 \Rightarrow 2$
- By counting the degrees of vertices, $2 \Rightarrow 3$

Paths and spanning trees

- Dijkstra's Algorithm
- Chinese Postman Problem
- Project scheduling
- Critical path
- Trees
- Spanning trees

Spanning Tree

Definition

- A *subgraph* H of a graph G is a graph such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.
- H is a *spanning subgraph* if $V(H) = V(G)$
- A *spanning tree* is a spanning subgraph that is also a tree.

Note

- If an edge appears in a subgraph, then both endpoints must also be included in the subgraph
- If a vertex appears in a subgraph, any number of its incident edges may be included

Kruskal's Algorithm

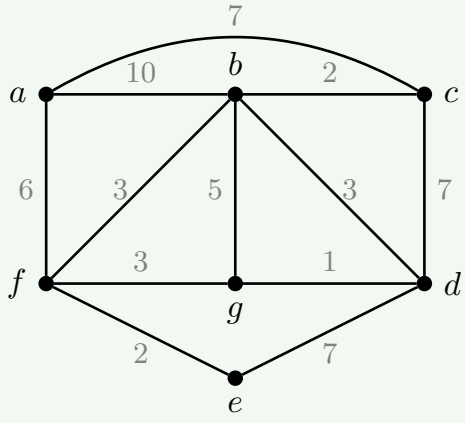
- First published in 1956 by Joseph Kruskal, an American mathematician
- Input: weighted connected graph $G = (V, E, \omega)$
- Steps
  1. Choose an unhighlighted edge of minimum weight. Highlight it and add it to $T = (V, E', \omega')$
  2. Repeat step 1 as long as no circuit is created.
- Output: minimum spanning tree (MST) T of G

Note

- If there are more than two edges with the least weight, randomly choose one
- At each step of the algorithm we are building a forest subgraph that will eventually result in a spanning tree

Kruskal's Algorithm – example

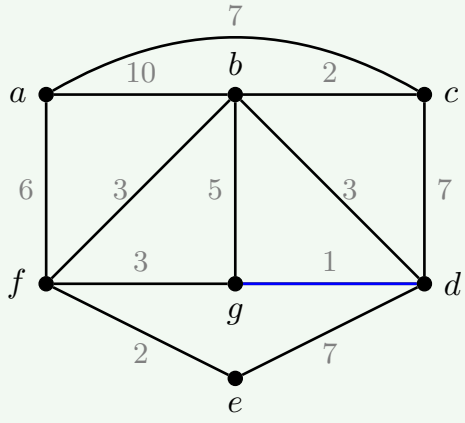
Example



- Step 1. edge with the least weight: gd

Kruskal's Algorithm – example

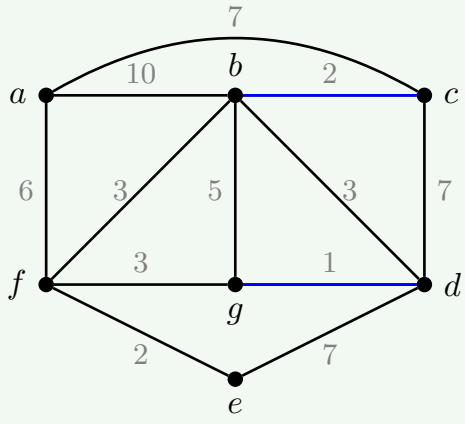
Example



- Step 1. two choices: ef and bc, let's choose bc

Kruskal's Algorithm – example

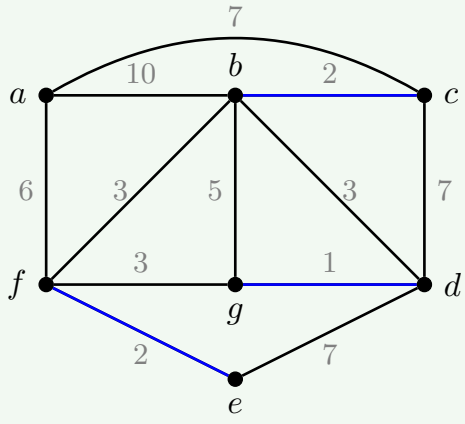
Example



- Step 1. the other edge with weight 2 is still a valid choice. Highlight ef

Kruskal's Algorithm – example

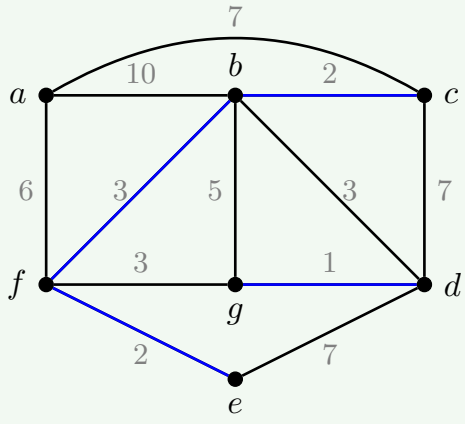
Example



- Step 1. the next smallest edge weight is 3. We randomly pick bf

Kruskal's Algorithm – example

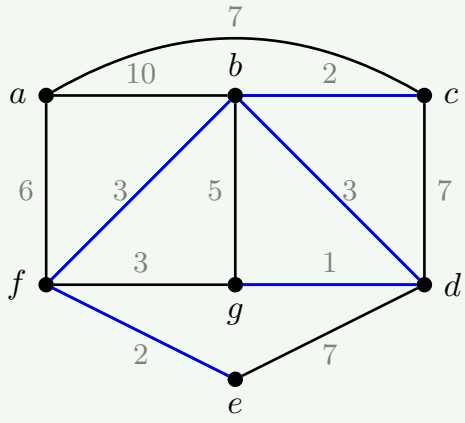
Example



- Step 1. both of the other edges of weight 3 are still available. We choose bd

Kruskal's Algorithm – example

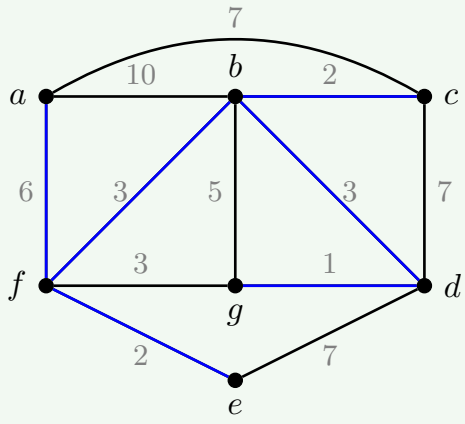
Example



- Step 1. cannot choose fg. The next smallest edge weight is 5. But we cannot choose bg. Next available edge is af of weight 6

Kruskal's Algorithm – example

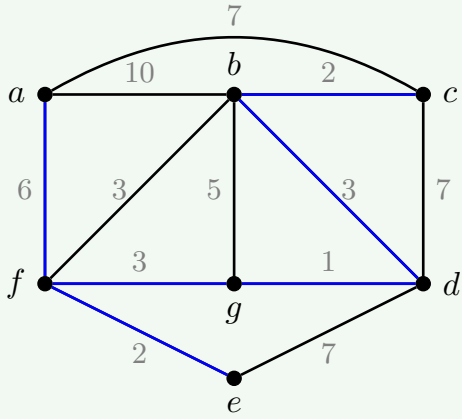
Example

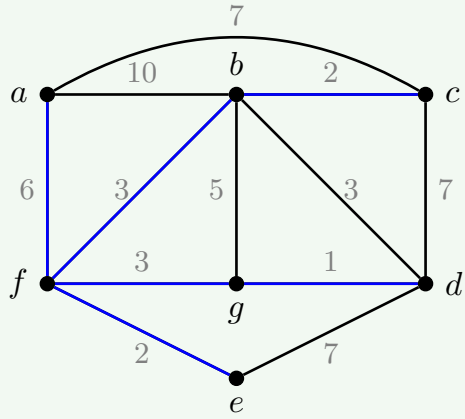


- Step 2. We now have a tree containing all vertices, so we can stop. Since the graph has 7 vertices and 6 edges, it is a tree. The MST has total weight 17.

Kruskal's Algorithm – example

Example





- When we were choosing an edge of weight 3, we could have made a different choice.

Kruskal's Algorithm – remarks

- When we skipped over an edge, e.g. bg of weight 5, we did so because including it would create a cycle
- This means a path between the endpoints of that edge, e.g. b and g, must already exist and the other edges along that path must each be of weight no greater than the edge we skip over
- In a way, if you think of finding a spanning tree as breaking cycles, then the largest edge on that cycle should never be chosen

Prim's Algorithm

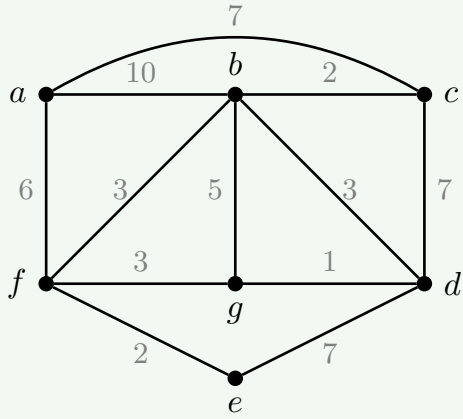
- Named after Robert C. Prim, American mathematician and computer scientist, published in 1957
- Originally discovered by Vojtěch Jarnik, Czech mathematician, in 1930
- Root: a clear starting point for the tree

Prim's Algorithm

- Input: Weighted connected graph $G = (V, E)$
- Steps
  1. Let v be the root. If no root is specified, choose a vertex at random. Highlight it and add it to $T = (V', E')$
  2. Among all edges incident to v, choose the one of minimum weight. Highlight it. Add the edge and its other endpoint to T.
  3. Let S be the set of all edges with exactly one endpoint in $V(T)$.
  4. Repeat step 3 until T contains all vertices of G
- Output: rooted MST T of G

Prim's Algorithm – example

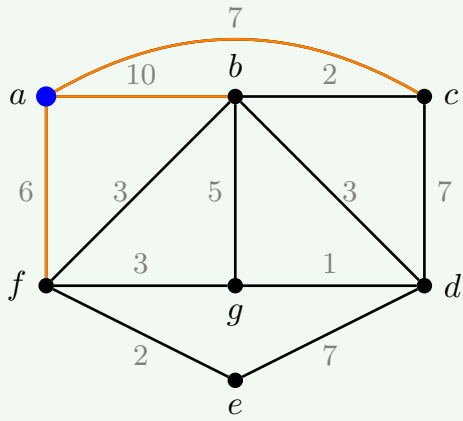
Example



- Same example as before
- Step 1. Choose a as the starting vertex

Prim's Algorithm – example

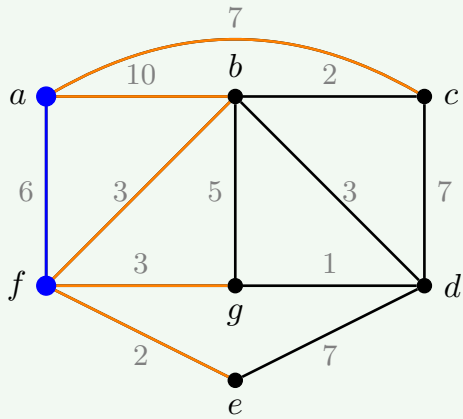
Example



- Step 2. among the edges incident to a, af, ab, ac, the edge of the least weight is af

Prim's Algorithm – example

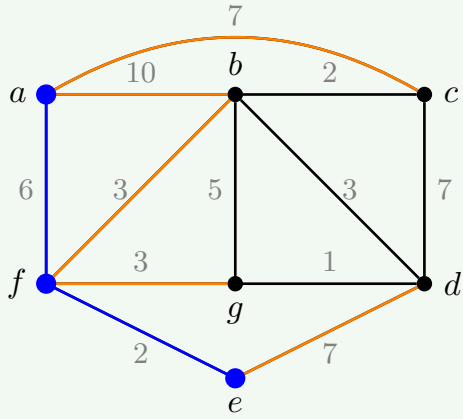
Example



- Step 3. S consists of edges with exactly one endpoint in the current tree, whose vertices are a and f. The edge with the minimum weight is ef

Prim's Algorithm – example

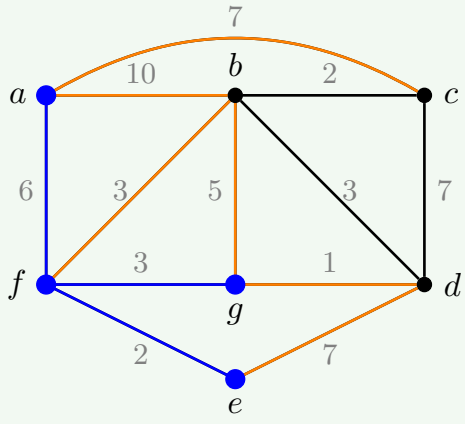
Example



- Step 3. S consists of edges with exactly one endpoint in the current tree, whose vertices are a, e, and f. The minimum-weight edges are fb and fg. Let us choose

Prim's Algorithm – example

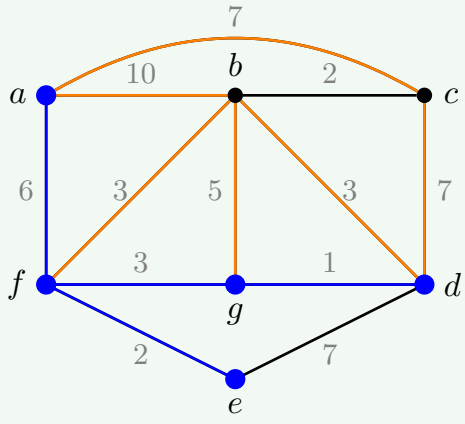
Example



- Step 3. The next edge to add to the tree is dg

Prim's Algorithm – example

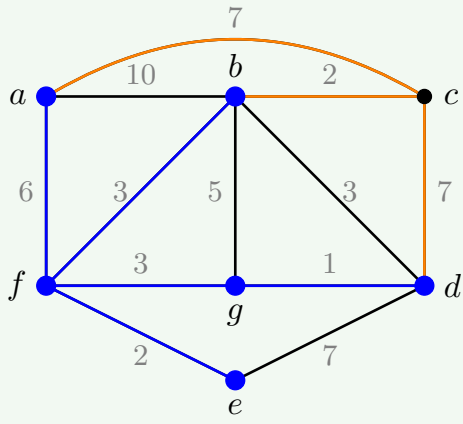
Example



- Step 3. There are two possible minimum weight edges, bf or bd. We choose bf

Prim's Algorithm – example

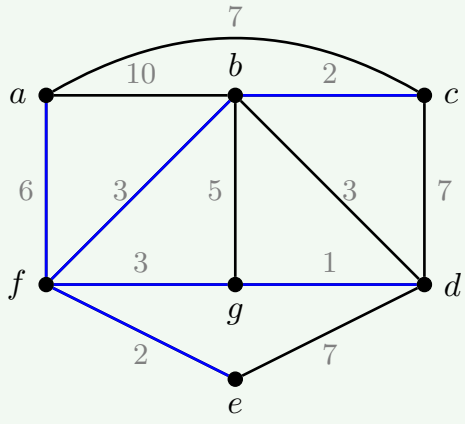
Example



- Step 3. Now the only edges to consider are those with one endpoint of c since this is the only vertex not part of our tree. The edge of minimum weight is bc

Prim's Algorithm – example

Example



- We have obtained a MST of weight 17