Algebra and Discrete Mathematics
ADM

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Course Outline

- Vectors and matrices
- System of linear equations
- Matrix inverse and determinants
- Vector spaces and matrix transformations
- Fundamental spaces and decompositions
- Eulerian tours
- Hamiltonian cycles
- Midterm
- Paths and spanning trees
- Trees and networks
- Matching

Recommended reading

- Anton, Howard, and Chris Rorres. Elementary linear algebra: applications version. John Wiley & Sons, 2013.
  - Sections 4.7, 4.8, 4.10, 5.1, 5.2, 9.1
  - Free copy online

Lecture outline

- Row space, column space, and null space
- Rank and nullity
- Eigenvalues and Eigenvectors
- Diagonalization
- LU-Decompositions
- Find LU-Decompositions

Fundamental spaces and decompositions

- Row space, column space, and null space
- Rank and nullity
- Eigenvalues and Eigenvectors
- Diagonalization
- LU-Decompositions
- Find LU-Decompositions

Definitions

Definition

Let $A \in \mathcal{M}_{m\times n}$

- *Row vectors* of A: rows of A
- *Row sapce* of A: the subspace of $\mathbb{R}^n$ spanned by the row vectors of A
- *Column vectors* of A: columns of A
- *Null space* of A: the solution space of $A\boldsymbol{x} = \mathbf{0}$

Elementary row operations

- $Ax = \mathbf{0}$ has augmented matrix $(A|\mathbf{0})$
- By performing row operations, we do not change the solution set of $Ax = \mathbf{0}$
- Thus

Theorem

Elementary row operations do not change the null space of a matrix.

By analyzing every elementary row operation, it can be shown that

Theorem

Elementary row operations do not change the row space of a matrix.

Find basis for null space

Example

$$A = \begin{pmatrix} 3 & 1 & 1 & 1 \\ 5 & -1 & 1 & -1 \end{pmatrix}$$

$$\begin{aligned} 3x_1 + x_2 + x_3 + x_4 &= 0 \\ 5x_1 - x_2 + x_3 - x_4 &= 0 \end{aligned}$$

The general solution in *vector form* is given by

$$\left(-\frac{2t}{7}, \quad -s - \frac{3t}{7}, \quad t, \quad s\right) = t\left(-\frac{2}{7}, \quad -\frac{3}{7}, \quad 1, \quad 0\right) + s\,(0, \quad -1, \quad 0, \quad 1)$$

A basis for the solution space, i.e. null space of A is

$$\left\{ \left(-\frac{2}{7}, \quad -\frac{3}{7}, \quad 1, \quad 0\right), \ (0, \quad -1, \quad 0, \quad 1) \right\}$$

Row space and column space of matrix in reduced row echelon form

Theorem

If a matrix R is in row echelon form, then the row vectors with the leading 1's (the nonzero row vectors) form a basis for the row space of R, and the pivot columns form a basis for the column space of R.

The theorem implies that

Remark

dimension of row space of $R =$ dimension of column space of $R =$ no of leading 1's

Row space and column space of matrix in reduced row echelon form

Example

$$R = \begin{pmatrix} 1 & -2 & 5 & 0 & 3 \\ 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

- Basis for row space:

$$\begin{pmatrix} 1, & -2, & 5, & 0, & 3 \end{pmatrix}, \begin{pmatrix} 0, & 1, & 3, & 0, & 0 \end{pmatrix}, \begin{pmatrix} 0, & 0, & 0, & 1, & 0 \end{pmatrix}$$

- Basis for column space:

$$\begin{pmatrix} 1, & 0, & 0, & 0 \end{pmatrix}, \begin{pmatrix} -2, & 1, & 0, & 0 \end{pmatrix}, \begin{pmatrix} 0, & 0, & 1, & 0 \end{pmatrix}$$

Find basis for row space

Note

Since elementary row operations do not change the row space, we can find a basis for row space by row reduction.

Example

$$A = \begin{pmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 2 & -6 & 9 & -1 & 8 & 2 \\ 2 & -6 & 9 & -1 & 9 & 7 \\ -1 & 3 & -4 & 2 & -5 & -4 \end{pmatrix} \xrightarrow{\text{row reduction}} R = \begin{pmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 0 & 0 & 1 & 3 & -2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

A basis for the row space of A:

$$\begin{pmatrix}1, & -3, & 4, & -2, & 5, & 4\end{pmatrix}, \begin{pmatrix}0, & 0, & 1, & 3, & -2, & -6\end{pmatrix}, \begin{pmatrix}0, & 0, & 0, & 0, & 1, & 5\end{pmatrix}$$

Elementary row operations and column spaces

By, e.g. analyzing each elementary operation:

- Elementary row operations do not alter dependence relationships or linear independence among the column vectors

Then we have

Theorem

If A, B are row equivalent matrices, then

- *A given set of column vectors of A is linearly independent iff the corresponding column vectors of B are linearly independent*
- *AA given set of column vectors of A forms a basis for the column space of A iff the corresponding column vectors of B forms a basis for the column space of B*

Basis for a column space by row reduction

Example

$$A = \begin{pmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 2 & -6 & 9 & -1 & 8 & 2 \\ 2 & -6 & 9 & -1 & 9 & 7 \\ -1 & 3 & -4 & 2 & -5 & -4 \end{pmatrix} \xrightarrow{\text{row reduction}} R = \begin{pmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 0 & 0 & 1 & 3 & -2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

A basis for the column space of A:

$$\begin{pmatrix} 1, & 2, & 2, & -1 \end{pmatrix}, \begin{pmatrix} 4, & 9, & 9, & -4 \end{pmatrix}, \begin{pmatrix} 5, & 8, & 9, & -5 \end{pmatrix}$$

Fundamental spaces and decompositions

- Row space, column space, and null space
- Rank and nullity
- Eigenvalues and Eigenvectors
- Diagonalization
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- Find LU-Decompositions

Rank

Theorem

The row space and the column space of A have the same dimension.

Proof.

- It follows from the previous discussions that elementary row operations do not change the dimension of the row space or of the column space of a matrix.
- Let R be a reduced row echelon form of A.
- We have shown that the row and column spaces of R have the same dimension.

□

Definition

The dimension of the row space (or column space) of a matrix A is called the *rank* of A, denoted by $\mathrm{rank}(A)$. The dimension of the null space of A is called the *nullity* of A, denoted by $\mathrm{nullity}(A)$

Rank and nullity – example

Example

$$A = \begin{pmatrix} -1 & 2 & 0 & 4 & 5 & -3 \\ 3 & -7 & 2 & 0 & 1 & 4 \\ 2 & -5 & 2 & 4 & 6 & 1 \\ 4 & -9 & 2 & -4 & -4 & 7 \end{pmatrix} \xrightarrow{\text{row reduction}} \begin{pmatrix} 1 & 0 & -4 & -28 & -37 & 13 \\ 0 & 1 & -2 & -12 & -16 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

- $\text{rank}(A) = 2$
- Nullity: dimension of solution space of $A\boldsymbol{x} = \mathbf{0}$

$$\begin{aligned} x_1 - 4x_3 - 28x_4 - 37x_5 + 13x_6 &= 0 \\ x_2 - 2x_3 - 12x_4 - 16x_5 + 5x_6 &= 0 \end{aligned}$$

Rank and nullity – example

Example

- $\text{rank}(A) = 2$
- Nullity: dimension of solution space of $A\boldsymbol{x} = \mathbf{0}$

$$\begin{aligned} x_1 - 4x_3 - 28x_4 - 37x_5 + 13x_6 &= 0 \\ x_2 - 2x_3 - 12x_4 - 16x_5 + 5x_6 &= 0 \end{aligned}$$

A vector form of the general solution is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = r \begin{pmatrix} 4 \\ 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} 28 \\ 12 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} 37 \\ 16 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + u \begin{pmatrix} -13 \\ -5 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$\text{nullity}(A) = 4$

Dimension theorem for matrices

Theorem

If A has n columns, then

$$\operatorname{rank}(A) + \operatorname{nullity}(A) = n$$

Proof.

number of leading variables $=$ number of leading 1's $= \operatorname{rank}(A)$

number of free variables in general solution of $A\boldsymbol{x} = \mathbf{0}$ $=$ number of parameters in the solution $= \operatorname{nullity}(A)$

number of leading variables $+$ number of free variables $= n$

□

Rank and nullity

From the proof we have established

Theorem

$A \in \mathcal{M}_{m\times n}$

- $\operatorname{rank}(A) =$ *the number of leading variables in the general solution of* $A\boldsymbol{x} = \mathbf{0}$
- $\operatorname{nullity}(A) =$ *the number of parameters in the general solution of* $A\boldsymbol{x} = \mathbf{0}$

Example

$A \in \mathcal{M}_{5\times 7}$

- $\operatorname{rank}(A) = 3$, find the number of parameters in the general solution of $A\boldsymbol{x} = \mathbf{0}$

$$\text{number of parameters} = \operatorname{nullity}(A) = 7 - 3 = 4$$

- $A\boldsymbol{x} = \mathbf{0}$ has a two-dimensional solution space, what is $\operatorname{rank}(A)$?

$$\operatorname{rank}(A) = 7 - \operatorname{nullity}(A) = 7 - 2 = 5$$

Dimension theorem for matrices – example

Example

$$A = \begin{pmatrix} -1 & 2 & 0 & 4 & 5 & -3 \\ 3 & -7 & 2 & 0 & 1 & 4 \\ 2 & -5 & 2 & 4 & 6 & 1 \\ 4 & -9 & 2 & -4 & -4 & 7 \end{pmatrix} \xrightarrow{\text{row reduction}} \begin{pmatrix} 1 & 0 & -4 & -28 & -37 & 13 \\ 0 & 1 & -2 & -12 & -16 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\operatorname{rank}(A) + \operatorname{nullity}(A) = 1 + 4 = 6$$

Orthogonal complement

Definition

If W is a subspace of $\mathbb{R}^n$, the the set of all vectors in $\mathbb{R}^n$ that are orthogonal to every vector in W is called the *orthogonal complement* of w, denoted by $W^\perp$, i.e.

$$W^\perp = \{\, \boldsymbol{v} \mid \boldsymbol{v} \in \mathbb{R}^n,\ \boldsymbol{v} \cdot \boldsymbol{w} = 0\ \forall \boldsymbol{w} \in W \,\}$$

Orthogonal complement

Theorem

If W is a subspace of $\mathbb{R}^n$, then

- $W^\perp$ *is a subspace of* $\mathbb{R}^n$
- $W \cap W^\perp = \{\mathbf{0}\}$
- $(W^\perp)^\perp = W$

Proof.

- Take any $\boldsymbol{u}, \boldsymbol{v} \in W^\perp$ and any $\alpha \in \mathbb{R}^n$. Then for any $\boldsymbol{w} \in W$

$$(\boldsymbol{u}+\boldsymbol{v}) \cdot \boldsymbol{w} = \boldsymbol{u} \cdot \boldsymbol{w} + \boldsymbol{v} \cdot \boldsymbol{w} = 0 + 0 = 0, \quad (\alpha \boldsymbol{u}) \cdot \boldsymbol{w} = \alpha(\boldsymbol{u} \cdot \boldsymbol{w}) = \alpha \times 0 = 0$$

  $W^\perp$ is closed under addition and scalar multiplication
- $W \cap W^\perp = \{\boldsymbol{x} \mid \boldsymbol{x} \cdot \boldsymbol{x} = 0\} = \{\mathbf{0}\}$
- We will only show $W \subseteq (W^\perp)^\perp$: take any $\boldsymbol{w} \in W$, we have $\boldsymbol{w} \cdot \boldsymbol{v} = 0$ for all $\boldsymbol{v} \in W^\perp$, thus $\boldsymbol{w} \in (W^\perp)^\perp$.

Orthogonal complement

Theorem

$A \in \mathcal{M}_{m\times n}$, the null space of A and the row space of A are orthogonal complements in $\mathbb{R}^n$.

Proof.

- Each solution of $A\boldsymbol{x} = \mathbf{0}$ satisfies $\boldsymbol{a}_i \cdot \boldsymbol{x} = \mathbf{0}$ for all i, where $\boldsymbol{a}_i$ is the ith row of A.
- The solution set of $A\boldsymbol{x} = \mathbf{0}$ consists of all vectors in $\mathbb{R}^n$ that are orthogonal to every row vector of A

□

Equivalent statements

Theorem

For any $A \in \mathcal{M}_{n\times n}$, the following statements are equivalent.

(a) *A is invertible*

(b) *$A\boldsymbol{x} = \mathbf{0}$ has only the trivial solution*

(c) *The reduced row echelon form of A is I_n*

(d) *A is expressible as a product of elementary matrices*

(e) *$A\boldsymbol{x} = \boldsymbol{b}$ is consistent $\forall \boldsymbol{b} \in \mathbb{R}^n$*

(f) *$A\boldsymbol{x} = \boldsymbol{b}$ has exactly one solution $\forall \boldsymbol{b} \in \mathbb{R}^n$*

(g) $\det(A) \neq 0$

(h) *The column vectors of A are linearly independent*

(i) *The row vectors of A are linearly independent*

(j) *The column vectors of A span $\mathbb{R}^n$*

(k) *The row vectors of A span $\mathbb{R}^n$*

(l) The column vectors of A form a basis for $\mathbb{R}^n$

(m) The row vectors of A form a basis for $\mathbb{R}^n$

(n) A has rank n

(o) A has nullity 0

(p) The orthogonal complement of the null space of A is $\mathbb{R}^n$

(q) The orthogonal complement of the row space of A is $\{\mathbf{0}\}$

Proof.

Let S be the set of row vectors of A, in the last lecture we have proved that

- S spans $\mathbb{R}^n$ iff $\det(A) \neq 0$
- S is a basis for $\mathbb{R}^n$ iff $\det(A) \neq 0$
- S is linearly independent iff $\det(A) \neq 0$

Since $\det(A) = \det(A^\top)$ and the row vectors of A are the column vectors of A, we have the equivalence of (g)(h), (g)(i), (g)(j), (g)(k), (g)(l), (g)(m), gives the equivalence from (a)-(m)

Equivalence statements

(b) $A\boldsymbol{x} = \mathbf{0}$ has only the trivial solution

(n) A has rank n

(o) A has nullity 0

Proof.

$\operatorname{rank}(A) + \operatorname{nullity}(A) = n$ proves the equivalence of (n)(o)
$(b) \Rightarrow (o)$ If $A\boldsymbol{x} = \mathbf{0}$ has only the trivial solution, then the null space is the zero space
$(o) \Rightarrow (b)$ We have proved that $\operatorname{nullity}(A) =$the number of parameters in the general solution of $A\boldsymbol{x} = \mathbf{0}$. $\operatorname{nullity}(A) = 0$ implies there are no free variables and the trivial solution is the only solution.
Now we have equivalence (a)-(o)

Equivalent statements

(k) The row vectors of A span $\mathbb{R}^n$

(p) The orthogonal complement of the null space of A is $\mathbb{R}^n$

(q) The orthogonal complement of the row space of A is $\{\mathbf{0}\}$

Proof.

We have just proved "the null space of A and the row space of A are orthogonal complements in $\mathbb{R}^n$," which shows the equivalence of (k)(p) and (k)(q) □

Kernel and range of T_A

Definition

$T_A : \mathbb{R}^n \to \mathbb{R}^m$ is a matrix transformation, the *kernel* of T_A, denoted $\ker(T_A)$, is the null space of A. The *range* of T_A, denoted $\mathrm{R}(T_A)$, is the column space of A

- $\ker(T_A)$: the set of all vectors in $\mathbb{R}^n$ that T_A maps into $\mathbf{0}$
- $\mathrm{R}(T_A)$: the set of all vectors in $\mathbb{R}^m$ that are images of at least one vector from $\mathbb{R}^n$

T_A and A

Theorem

$A \in \mathcal{M}_{n\times n}$, $T_A : \mathbb{R}^n \to \mathbb{R}^n$, *the following statements are equivalent*

(a) A *is invertible*

(b) $\ker(T_A) = \{\mathbf{0}\}$

(c) $\mathrm{R}\,(T_A) = \mathbb{R}^n$

(d) T_A *is bijective*

Proof.

$(a) \Leftrightarrow (b)$ $(a) \Leftrightarrow \text{nullity}\,(A) = 0$
$(a) \Leftrightarrow (c)$ $(a) \Leftrightarrow$ column vectors of A span $\mathbb{R}^n$
$(a) \Rightarrow (d)$ $(a) \Rightarrow$ for any $\boldsymbol{b} \in \mathbb{R}^n$, $A\boldsymbol{x} = \boldsymbol{b}$ has a solution$\Rightarrow$ T_A is injective. If T_A is not injective, $\exists \boldsymbol{v}_1, \boldsymbol{v}_2 \in \mathbb{R}^n$ s.t.

$$T_A(\boldsymbol{v}_1) = T_A(\boldsymbol{v}_2) \Rightarrow T_A(\boldsymbol{v}_1 - \boldsymbol{v}_2) = \mathbf{0} \Rightarrow \text{nullity}\,(A) \neq 0$$

a contradiction. Thus, T_A is injective.

T_A and A

Theorem

$A \in \mathcal{M}_{n\times n}$, $T_A : \mathbb{R}^n \to \mathbb{R}^n$, *the following statements are equivalent*

(a) A *is invertible*

(b) $\ker(T_A) = \{\mathbf{0}\}$

(c) $\mathrm{R}\,(T_A) = \mathbb{R}^n$

(d) T_A *is bijective*

Proof.

$(d) \Rightarrow (a)$ T_A is surjective $\Rightarrow$ for any $\boldsymbol{b} \in \mathbb{R}^n$, $A\boldsymbol{x} = \boldsymbol{b}$ has a solution $\Rightarrow (a)$ □

Remark

With a bit modification of the proof, we can prove that

$$T_A \text{ is surjective} \Leftrightarrow T_A \text{ is injective} \Leftrightarrow T_A \text{ is bijective}$$

The rotation operator on RR^2 is surjective

Example

- T : Rotation about the origin through ab angle θ

$$[T] = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

- $\det([T]) = 1 \neq 0$

Fundamental spaces and decompositions

- Row space, column space, and null space
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- Find LU-Decompositions

Definition

Definition

Let $A \in \mathcal{M}_{n \times n}$, a nonzero $\boldsymbol{x} \in \mathbb{R}^n$ is called an *eigenvector* of A (or of the matrix operator T_A) if $A\boldsymbol{x}$ is a scalar multiple of $\boldsymbol{x}$, i.e.

$$A\boldsymbol{x} = \lambda \boldsymbol{x}$$

for some $\lambda \in \mathbb{R}$. λ is called an *eigenvalue* of A (or T_A) and $\boldsymbol{x}$ is said to be an *eigenvector corresponding to* λ.

- In general $A\boldsymbol{x}$ differs from $\boldsymbol{x}$ in both magnitude and direction
- For eigenvectors, the direction is unchanged

Eigenvector – example

Example

$$\begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$$

$\boldsymbol{x} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is an eigenvector of A corresponding to $\lambda = 3$

$$A\boldsymbol{x} = 3\boldsymbol{x}$$

Characteristic equation

- $A\boldsymbol{x} = \lambda\boldsymbol{x}$ can be rewritten as $A\boldsymbol{x} = \lambda I\boldsymbol{x}$ or

$$(\lambda I - A)\boldsymbol{x} = \mathbf{0}$$

- For λ to be an eigenvalue of A, the equation must have a **nonzero** solution for $\boldsymbol{x}$
- Which is true iff $\det(\lambda I - A) = 0$

Theorem

Let $A \in \mathcal{M}_{n\times n}$, λ is an eigenvalue of A iff

$$\det(\lambda I - A) = 0$$

This is called the characteristic equation of A.

Finding eigenvalues – example

Example

Consider $A = \begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix}$, $\det(\lambda I - A) = 0$ gives

$$\begin{vmatrix} \lambda - 3 & 0 \\ -8 & \lambda + 1 \end{vmatrix} = 0 \implies (\lambda - 3)(\lambda + 1) = 0$$

Thus, the eigenvalues of A are 3 and -1.

Characteristic polynomial

$$\det(\lambda I - A) = \begin{vmatrix} \lambda - a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & \lambda - a_{22} & \cdots & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n2} & \cdots & \lambda - a_{nn} \end{vmatrix}$$

- With cofactor expansion, the highest power of λ appears when multiplying all diagonal entries
- Characteristic equation of A takes the form

$$\lambda^n + \alpha_1 \lambda^{n-1} + \cdots + \alpha_n = 0$$

- The polynomial (left side of the equation)

$$p(\lambda) = \lambda^n + \alpha_1 \lambda^{n-1} + \cdots + \alpha_n$$

  is called the *characteristic polynomial* of A
- Degree n polynomial has at most n distinct roots $\Longrightarrow$ at most n distinct eigenvalues

Characteristic polynomial – example

Example

$A = \begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix}$ has characteristic polynomial

$$p(\lambda) = (\lambda - 3)(\lambda + 1) = \lambda^2 - 2\lambda - 3$$

Characteristic polynomial – example

Example

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{pmatrix}$$

Characteristic polynomial of A is

$$\det(\lambda I - A) = \begin{vmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ -4 & 17 & \lambda - 8 \end{vmatrix} = \lambda^3 - 8\lambda^2 + 17\lambda - 4$$

- First we observe that an integer solution (if any) of a polynomial equation with integer coefficients must be a divisor of the constant term
- In our case, divisors of -4: ± 1, ± 2, ± 4
- $\lambda = 4$ is a solution

$$(\lambda - 4)(\lambda^2 - 4\lambda + 1) = 0$$

Characteristic polynomial – example

Example

Characteristic polynomial of A is $\lambda^3 - 8\lambda^2 + 17\lambda - 4$

- $\lambda = 4$ is a solution

$$(\lambda - 4)(\lambda^2 - 4\lambda + 1) = 0$$

- Solve the quadratic equation by the quadratic formula

$$\lambda = \frac{4 \pm \sqrt{4^2 - 4}}{2} = 2 \pm \sqrt{3}$$

- Eigenvalues of A are

$$\lambda_1 = 4, \quad \lambda_2 = 2 + \sqrt{3}, \quad \lambda_3 = 2 - \sqrt{3}$$

Eigenvalues of an upper triangular matrix

Example

$A = \begin{pmatrix} 1 & 8 & 0 \\ 0 & 2 & 6 \\ 0 & 0 & 3 \end{pmatrix}$, the characteristic equation of A is

$$\det(\lambda I - A) = \begin{vmatrix} \lambda - 1 & 8 & 0 \\ 0 & \lambda - 2 & 6 \\ 0 & 0 & \lambda - 3 \end{vmatrix} = (\lambda - 1)(\lambda - 2)(\lambda - 3)$$

Eigenvalues of triangular matrices

Theorem

If $A \in \mathcal{M}_{n\times n}$ is a triangular matrix (upper triangular, lower triangular, diagonal), then the eigenvalues of A are the entries on the main diagonal of A.

Equivalent statements

Theorem

Let $A \in \mathcal{M}_{n\times n}$*, the following statements are equivalent*

- λ *is an eigenvalue of* A
- λ *is a solution of the characteristic equation* $\det(\lambda I - A) = 0$
- *The system of equations* $(\lambda I - A)\boldsymbol{x} = \mathbf{0}$ *has nontrivial solutions*
- *There is a nonzero vector* $\boldsymbol{x}$ *s.t.* $A\boldsymbol{x} = \lambda\boldsymbol{x}$

Eigenspace

- Eigenvectors of A corresponding to an eigenvalue λ are the **nonzero** vectors that satisfy
$$(\lambda I - A)\boldsymbol{x} = \mathbf{0}$$
- The solution space is called the *eigenspace* of A corresponding to λ consist of eigenvectors of A corresponding to λ and $\mathbf{0}$
- Eigenspace can also be reviewed as
  - the null space of the matrix $\lambda I - A$
  - the kernel of the matrix operator $T_{\lambda I - A} : \mathbb{R}^n \to \mathbb{R}^n$
  - the set of vectors for which $A\boldsymbol{x} = \lambda \boldsymbol{x}$

Find bases for eigenspaces – example

Example

$$A = \begin{pmatrix} -1 & 3 \\ 2 & 0 \end{pmatrix}, \quad \begin{vmatrix} \lambda + 1 & -3 \\ -2 & \lambda \end{vmatrix} = (\lambda - 2)(\lambda + 3) = 0$$

Eigenvalues of A are $\lambda_1 = 2$, $\lambda_2 = 3$.

$$\begin{pmatrix} -3 & -3 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Longrightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = t \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Thus the eigenspace corresponding to $\lambda_1 = 2$ has basis $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ The eigenspace corresponding to $\lambda_2 = 3$ has basis $\begin{pmatrix} -\frac{3}{2} \\ 1 \end{pmatrix}$

Equivalent statements

Theorem

For any $A \in \mathcal{M}_{n \times n}$, the following statements are equivalent.

(a) A *is invertible*
(b) $A\boldsymbol{x} = \mathbf{0}$ *has only the trivial solution*
(c) *The reduced row echelon form of* A *is* I_n
(d) A *is expressible as a product of elementary matrices*
(e) $A\boldsymbol{x} = \boldsymbol{b}$ *is consistent* $\forall \boldsymbol{b} \in \mathbb{R}^n$
(f) $A\boldsymbol{x} = \boldsymbol{b}$ *has exactly one solution* $\forall \boldsymbol{b} \in \mathbb{R}^n$
(g) $\det(A) \neq 0$
(h) *The column vectors of* A *are linearly independent*
(i) *The row vectors of* A *are linearly independent*
(j) *The column vectors of* A *span* $\mathbb{R}^n$
(k) *The row vectors of* A *span* $\mathbb{R}^n$

(l) The column vectors of A form a basis for $\mathbb{R}^n$

(m) The row vectors of A form a basis for $\mathbb{R}^n$

(n) A has rank n

(o) A has nullity 0

(p) The orthogonal complement of the null space of A is $\mathbb{R}^n$

(q) The orthogonal complement of the row space of A is $\{\mathbf{0}\}$

(r) $\ker(T_A) = \{\mathbf{0}\}$

(s) $\mathrm{R}\,(T_A) = \{\mathbb{R}^n\}$

(t) T_A is surjective

(u) $\lambda = 0$ is not an eigenvalue of A

Proof.

The equivalence of (a)(r)(s)(t) was proved just now

We will prove $(g) \Leftrightarrow (u)$

Equivalent statements

(g) $\det(A) \neq 0$

(u) $\lambda = 0$ is not an eigenvalue of A

Proof.

We will prove $(g) \Leftrightarrow (u)$

$\lambda = 0$ is a solution of the characteristic equation

$$\lambda^n + \alpha_1 \lambda^{n-1} + \cdots + \alpha_n = 0$$

iff $\alpha_n = 0$. On the other hand, setting $\lambda = 0$

$$\det(\lambda I - A) = \det(-A) = \lambda^n + \alpha_1 \lambda^{n-1} + \cdots + \alpha_n = \alpha_n$$

i.e. $(-1)^n \det(A) = \alpha_n$. □

Fundamental spaces and decompositions

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Matrix diagonalization problem

- $A, P \in \mathcal{M}_{n\times n}$, P invertible
- *similarity transformation*: $A \mapsto P^{-1}AP$

$$\det(P^{-1}AP) = \det(A)$$

  i.e. similarity transformation preserves determinant
- Any property that is preserved by a similarity transformation is called a *similarity invariant* and is said to be *invariant under similarity*

Similarity invariants

Determinant	$\det(A) = \det(P^{-1}AP)$
Invertibility	A is invertible iff $P^{-1}AP$ is invertible
Rank	$\operatorname{rank}(A) = \operatorname{rank}\left(P^{-1}AP\right)$
Nullity	$\operatorname{nullity}(A) = \operatorname{nullity}\left(P^{-1}AP\right)$
Trace	$\operatorname{tr}(A) = \operatorname{tr}\left(P^{-1}AP\right)$
Characteristic polynomial	
Eigenvalues	
Eigenspace dimension	Eigenspace of A corresponding to λ has same dimension as eigenspace of $P^{-1}AP$ corresponding to λ

Similar matrices

Definition

$A, B \in \mathcal{M}_{n\times n}$, if $\exists P$ invertible s.t. $B = P^{-1}AP$, then we say B *is similar to* A.

- If $B = P^{-1}AP$, let $Q = P^{-1}$
- $A = Q^{-1}BQ$
- We usually say that A and B are similar matrices

Diagonalizable

Definition

$A \in \mathcal{M}_{n\times n}$ is said to be diagonalizable if it is similar to some diagonal matrix, i.e. $A = P^{-1}DP$ for a diagonal matrix D. P is said to *diagonalize* A.

Diagonalization

- Find n linearly independent eigenvectors of A: $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$
- Construct
$$P = \begin{pmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \cdots & \boldsymbol{v}_n \end{pmatrix}$$
- Then
$$AP = \begin{pmatrix} A\boldsymbol{v}_1 & A\boldsymbol{v}_2 & \cdots & A\boldsymbol{v}_n \end{pmatrix} = \begin{pmatrix} \lambda_1\boldsymbol{v}_1 & \lambda_2\boldsymbol{v}_2 & \cdots & \lambda_n\boldsymbol{v}_n \end{pmatrix} = PD,$$
where D is the diagonal matrix that has $\lambda_1, \lambda_2, \ldots, \lambda_n$
- Since $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$ are linearly independent, P is invertible, we have $P^{-1}AP = D$

Diagonalization – example

Example

$$A = \begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix}$$

Characteristic equation of A is $(\lambda - 1)(\lambda - 2)^2 = 0$
Bases for the eigenspace

$$\lambda_1 = 2 : \boldsymbol{v}_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \boldsymbol{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}; \quad \lambda_2 = 1 : \boldsymbol{v}_3 = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$$

$$P = \begin{pmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}, \quad P^{-1}AP = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Diagonalizable matrices

Theorem

$A \in \mathcal{M}_{n\times n}$, A is diagonalizable iff A has n linearly independent eigenvectors.

Proof.

We have just proved $\Leftarrow$.

$\Rightarrow$ Assume $AP = PD$, P has columns $\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_n$, D has diagonal entries $\lambda_1, \lambda_2, \dots, \lambda_n$

$$AP = \begin{pmatrix} A\boldsymbol{v}_1 & A\boldsymbol{v}_2 & \cdots & A\boldsymbol{v}_n \end{pmatrix}, \quad PD = \begin{pmatrix} \lambda_1\boldsymbol{v}_1 & \lambda_2\boldsymbol{v}_2 & \cdots \lambda_n\boldsymbol{v}_n \end{pmatrix}$$

P invertible implies that $\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_n$ are linearly independent. □

Linearly independent eigenvectors

Theorem

- *If* $\lambda_1, \lambda_2, \ldots, \lambda_k$ *are distinct eigenvalues of* A*, and if* $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k$ *are corresponding eigenvectors, then* $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k\}$ *is a linearly independent set*
- *If* $\lambda_1, \lambda_2, \ldots, \lambda_k$ *are distinct eigenvalues of* A*, and if* $S_1, S_2, \ldots, S_k$ *are corresponding sets of linearly independent eigenvectors, then the union of these sets is linearly independent.*
- *An* $n \times n$ *matrix with* n *distinct eigenvalues is diagonalizable.*

Fundamental spaces and decompositions

- Row space, column space, and null space
- Rank and nullity
- Eigenvalues and Eigenvectors
- Diagonalization
- LU-Decompositions
- Find LU-Decompositions

Solving linear systems

- Gaussian elimination (reduction to row echelon form) and Gauss–Jordan elimination (reduction to reduced row echelon form)
- Fine for small-scale problems, not suitable for large-scale problems in which computer roundoff error, memory usage, and speed are concerns
- We will discuss a method for solving linear systems based on factoring its coefficient matrix into a product of lower and upper triangular matrices
- LU-decomposition is the basis for many computer algorithms in common use

LU-decomposition

Definition

A factorization of a square matrix A as

$$A = LU,$$

where L is lower triangular and U is upper triangular, is called an *LU-decomposition* (or *LU-factorization*) of A.

Solving linear systems by LU-decomposition

Step 1. Rewrite the system $A\boldsymbol{x} = \boldsymbol{b}$ as

$$LU\boldsymbol{x} = \boldsymbol{b} \tag{1}$$

Step 2. Define a new $n \times 1$ matrix $\boldsymbol{y}$ by

$$U\boldsymbol{x} = \boldsymbol{y} \tag{2}$$

Step 3. Rewrite (1) as

$$L\boldsymbol{y} = \boldsymbol{b}$$

and solve this system for $\boldsymbol{y}$

Step 4. Substitute $\boldsymbol{y}$ in (2) and sovle for $\boldsymbol{x}$.

Solving linear systems by LU-decomposition

Example

$A = LU$

$$\begin{pmatrix} 2 & 6 & 2 \\ -3 & -8 & 0 \\ 4 & 9 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ -3 & 1 & 0 \\ 4 & -3 & 7 \end{pmatrix} \begin{pmatrix} 1 & 3 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$$

$A\boldsymbol{x} = \boldsymbol{b}$

$$\begin{pmatrix} 2 & 6 & 2 \\ -3 & -8 & 0 \\ 4 & 9 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix}$$

$LU\boldsymbol{x} = \boldsymbol{b}$

$$\begin{pmatrix} 2 & 0 & 0 \\ -3 & 1 & 0 \\ 4 & -3 & 7 \end{pmatrix} \begin{pmatrix} 1 & 3 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix}$$

Solving linear systems by LU-decomposition

Example

$U\boldsymbol{x} = \boldsymbol{y}$ and $L\boldsymbol{y} = \boldsymbol{b}$

$$\begin{pmatrix} 1 & 3 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}, \quad \begin{pmatrix} 2 & 0 & 0 \\ -3 & 1 & 0 \\ 4 & -3 & 7 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix}$$

Solve for $\boldsymbol{y}$

$$\begin{aligned} 2y_1 &= 2 \\ -3y_1 + y_2 &= 2 \\ 4y_1 - 3y_2 + 7y_3 &= 3 \end{aligned} \Longrightarrow y_1 = 1, \quad y_2 = 5, \quad y_3 = 2$$

Substitute $\boldsymbol{y}$ to $U\boldsymbol{x} = \boldsymbol{y}$ and solve for $\boldsymbol{x}$

$$\begin{pmatrix} 1 & 3 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \\ 2 \end{pmatrix} \Longrightarrow x_1 = 2, \quad x_2 = -1, \quad x_3 = 2$$

Solving linear systems by LU-decomposition

- Consider $A = LU \in \mathcal{M}_{3\times 3}$
- $L\boldsymbol{y} = \boldsymbol{b}$

$$\begin{pmatrix} \ell_{11} & 0 & 0 \\ \ell_{21} & \ell_{22} & 0 \\ \ell_{31} & \ell_{32} & \ell_{33} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

- Solving for $\boldsymbol{y}$

$$\begin{aligned} \ell_{11}y_1 &= b_1 \\ \ell_{21}y_1 + \ell_{22}y_2 &= b_2 \\ \ell_{31}y_1 + \ell_{32}y_2 + \ell_{33}y_3 &= b_3 \end{aligned} \implies \begin{aligned} y_1 &= \frac{b_1}{\ell_{11}} \\ y_2 &= \frac{1}{\ell_{22}}(b_2 - \ell_{21}y_1) \\ y_3 &= \frac{1}{\ell_{33}}(b_3 - \ell_{31}y_1 - \ell_{32}y_2) \end{aligned}$$

- General formula

$$y_k = \frac{1}{\ell_{kk}}\left(b_k - \sum_{s=1}^{k-1} \ell_{ks}y_s\right)$$

Solving linear systems by LU-decomposition

- Consider $A = LU \in \mathcal{M}_{3\times 3}$
- $U\boldsymbol{x} = \boldsymbol{y}$

$$\begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

- Solving for $\boldsymbol{x}$
- General formula

$$x_k = \frac{1}{u_{kk}} \left(b_k - \sum_{s=k+1}^{n} u_{ks} x_s \right)$$

A sufficient condition

- Not every square matrix has an LU-decomposition

Theorem

If A is a square matrix that can be reduced to a row echelon form U by Gaussian elimination without row interchanges, then A can be factored as $A = LU$, where L is a lower triangular matrix.

Proof.

- U: row echelon form, upper triangular
- Row operations on A can be accomplished by multiplying A on the left by an appropriate sequence of elementary matrices

$$E_k \cdots E_2 E_1 A = U$$

- Elementary matrices are invertible

$$A = E_1^{-1} E_2^{-1} \cdots E_k^{-1} U$$

A sufficient condition

Proof.

$$A = E_1^{-1} E_2^{-1} \cdots E_k^{-1} U$$

- U: row echelon form, upper triangular
- $L := E_1^{-1} E_2^{-1} \cdots E_k^{-1}$
- E_j is lower triangular $\xrightarrow{\text{Tutorial 4}}$ E_j^{-1} is lower triangular $\xrightarrow{\text{Tutorial 1}}$ L is lower triangular
- Since row interchanges are excluded, each E_j results by
  - adding a scalar multiple of one row of an identity matrix to a row below
  - multiplying one row of an identity matrix by a nonzero scalar

□

Fundamental spaces and decompositions

- Row space, column space, and null space
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- Diagonalization
- LU-Decompositions
- Find LU-Decompositions

Find LU-decomposition – example

Example

$$A = \begin{pmatrix} 2 & 6 & 2 \\ -3 & -8 & 0 \\ 4 & 9 & 2 \end{pmatrix}$$

To obtain LU-decomposition, we reduce A to a row echelon form U using Gaussian elimination

Step 1 $\xrightarrow{R_1 \to \frac{1}{2} R_1} \begin{pmatrix} 1 & 3 & 1 \\ -3 & -8 & 0 \\ 4 & 9 & 2 \end{pmatrix}$ $E_1 = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $E_1^{-1} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Step 2 $\xrightarrow{R_2 \to 3R_1 + R2} \begin{pmatrix} 1 & 3 & 1 \\ 0 & 1 & 3 \\ 4 & 9 & 2 \end{pmatrix}$ $E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $E_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Find LU-decomposition – example

Example

Step 3 $\xrightarrow{R_3 \to -4R_1 + R_3} \begin{pmatrix} 1 & 3 & 1 \\ 0 & 1 & 3 \\ 0 & -3 & -2 \end{pmatrix}$ $\quad E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{pmatrix}$ $\quad E_3^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{pmatrix}$

Step 4 $\xrightarrow{R_3 \to 3R_2 + R_3} \begin{pmatrix} 1 & 3 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 7 \end{pmatrix}$ $\quad E_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{pmatrix}$ $\quad E_4^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{pmatrix}$

Step 5 $\xrightarrow{R_3 \to \frac{1}{7} R_3} \begin{pmatrix} 1 & 3 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$ $\quad E_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & \frac{1}{7} \end{pmatrix}$ $\quad E_4^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 7 \end{pmatrix}$

$$L = E_1^{-1} E_2^{-1} E_3^{-1} E_4^{-1} = \begin{pmatrix} 2 & 0 & 0 \\ -3 & 1 & 0 \\ 4 & -3 & 7 \end{pmatrix}$$

Find LU-decomposition

Example

$$L = \begin{pmatrix} 2 & 0 & 0 \\ -3 & 1 & 0 \\ 4 & -3 & 7 \end{pmatrix}$$

- $\ell_{11} = 2$: a multiplier of $\frac{1}{2}$ was needed in Step 1 to introduce a leading 1 in the first row
- $\ell_{33} = 7$: a multiplier of $\frac{1}{7}$ was needed in Step 5 to introduce a leading 1 in the third row
- $\ell_{22} = 1$: a multiplier of 1 to introduce a leading 1 in the second row
- To introduce 0 below the leading 1's:
- $\ell_{21} = -3$: Step 2 $R_2 \to 3R_1 + R2$
- $\ell_{31} = 4$: Step 3 $R_3 \to -4R_1 + R_3$
- $\ell_{32=-3}$: Step 4 $R_3 \to 3R_2 + R_3$

Find LU-decomposition

- Each position along the main diagonal of L: reciprocal of the multiplier that introduced the leading 1 in that position in U
- Each position below the main diagonal of L: the negative of the multiplier used to introduce the zero in that position in U

Find LU-decomposition – example

Example

$$A = \begin{pmatrix} 6 & -2 & 0 \\ 9 & -1 & 1 \\ 3 & 7 & 5 \end{pmatrix}$$

$$\xrightarrow{R_1 \to \frac{1}{6} R_1} \begin{pmatrix} 1 & -\frac{1}{3} & 0 \\ 9 & -1 & 1 \\ 3 & 7 & 5 \end{pmatrix} \quad \text{multiplier } \frac{1}{6}\text{, first row} \quad \begin{pmatrix} 6 & 0 & 0 \\ * & * & 0 \\ * & * & * \end{pmatrix}$$

$$\xrightarrow[R_3 - 3R_1]{R_2 \to -9R_1 + R_2} \begin{pmatrix} 1 & -\frac{1}{3} & 0 \\ 0 & 2 & 1 \\ 0 & 8 & 5 \end{pmatrix} \quad \begin{matrix} \text{multiplier } -9\text{, position } (2,1), \\ \text{multiplier } -3\text{, position } (3,1) \end{matrix} \quad \begin{pmatrix} 6 & 0 & 0 \\ 9 & * & 0 \\ 3 & * & * \end{pmatrix}$$

Find LU-decomposition – example

Example

$$\xrightarrow{R_2\to -\frac{1}{2}R_2} \begin{pmatrix} 1 & -\frac{1}{3} & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 8 & 5 \end{pmatrix}$$ multiplier $\frac{1}{2}$, second row $$\begin{pmatrix} 6 & 0 & 0 \\ 9 & 2 & 0 \\ 3 & * & * \end{pmatrix}$$

$$\xrightarrow{R_3\to -8R_2+R_3} \begin{pmatrix} 1 & -\frac{1}{3} & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix}$$ multiplier -8, position $(3,2)$ $$\begin{pmatrix} 6 & 0 & 0 \\ 9 & 2 & 0 \\ 3 & 8 & * \end{pmatrix}$$

$$U = \begin{pmatrix} 1 & -\frac{1}{3} & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix}$$ multiplier 1, third row $$L = \begin{pmatrix} 6 & 0 & 0 \\ 9 & 2 & 0 \\ 3 & 8 & 1 \end{pmatrix}$$

Another sufficient condition

Definition

$A = (a_{ij}) \in \mathcal{M}_{n\times n}$, the *leading principal minors* of A are $\det(A_1), \det(A_2), \ldots, \det(A_n)$, where A_k is the top-left $k \times k$ submatrix of A

$$A_k = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kk} \end{pmatrix}$$

Theorem

Let $A = (a_{ij}) \in \mathcal{M}_{n\times n}$, if the leading principal minors of A are all nonzero, then there exists an LU-decomposition $A = LU$.

Remark

If A is invertible, then the sufficient condition is also necessary.

Find LU-decomposition – Doolittle decomposition

Example

$$A = \begin{pmatrix} 2 & 5 & 6 \\ 4 & 13 & 19 \\ 6 & 27 & 50 \end{pmatrix} = LU, \quad L = \begin{pmatrix} 1 & 0 & 0 \\ \ell_{21} & 1 & 0 \\ \ell_{31} & \ell_{32} & 1 \end{pmatrix}, \quad U = \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix}$$

First row

$$u_{11} = 2, \quad u_{12} = 5, \quad u_{13} = 6$$

Second row

$$\begin{aligned} \ell_{21}u_{11} &= 4 \\ \ell_{21}u_{12} + u_{22} &= 13 \\ \ell_{21}u_{13} + u_{23} &= 19 \end{aligned} \implies \begin{aligned} \ell_{21} &= 2 \\ u_{22} &= 12 - 2 \times 5 = 3 \\ u_{23} &= 19 - 2 \times 6 = 7 \end{aligned}$$

Third row

$$\begin{aligned} \ell_{31}u_{11} &= 6 \\ \ell_{31}u_{12} + \ell_{32}u_{22} &= 27 \\ \ell_{31}u_{13} + \ell_{32}u_{23} + u_{33} &= 50 \end{aligned} \implies \begin{aligned} \ell_{31} &= 3 \\ \ell_{32} &= \frac{27 - 3 \times 5}{3} = 4 \\ u_{33} &= 50 - 3 \times 6 - 4 \times 7 = 4 \end{aligned}$$

Find LU-decomposition – Doolittle decomposition

$$L = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ \ell_{21} & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ \ell_{n1} & \ell_{n2} & \ell_{n3} & \cdots & 1 \end{pmatrix}, \quad U = \begin{pmatrix} u_{11} & u_{12} & u_{13} & \cdots & u_{1n} \\ 0 & u_{22} & u_{23} & \cdots & u_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & u_{nn} \end{pmatrix}.$$

For reach row i

- Computing U: for $j \geq i$:

$$u_{ij} = a_{ij} - \sum_{k=1}^{i-1} \ell_{ik} u_{kj}. \tag{3}$$

- Computing L: for $j < i$:

$$\ell_{ij} = \frac{1}{u_{jj}} \left(a_{ij} - \sum_{k=1}^{i-1} \ell_{ik} u_{kj} \right). \tag{4}$$

Find LU-decomposition – Crout decomposition

$$L = \begin{pmatrix} \ell_{11} & 0 & 0 & \cdots & 0 \\ \ell_{21} & \ell_{22} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \ell_{n1} & \ell_{n2} & \ell_{n3} & \cdots & \ell_{nn} \end{pmatrix}, \quad U = \begin{pmatrix} 1 & u_{12} & u_{13} & \cdots & u_{1n} \\ 0 & 1 & u_{23} & \cdots & u_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

For each column:

- Computing L: for $i \geq j$:

$$\ell_{ij} = a_{ij} - \sum_{k=1}^{j-1} \ell_{ik} u_{kj}. \tag{5}$$

- Computing U: for $i > j$:

$$u_{ij} = \frac{1}{\ell_{jj}} \left(a_{ij} - \sum_{k=1}^{j-1} \ell_{jk} u_{ki} \right). \tag{6}$$

Same result as using Gaussian elimination

LU-dcomposition is not unique

Example

- Note that Crout decomposition and Doolittle decomposition produce different matrices
- Another example: $A = \begin{pmatrix} 2 & 10 \\ 7 & 44 \end{pmatrix}$

$$L = \begin{pmatrix} 1 & 0 \\ \frac{7}{2} & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 2 & 10 \\ 0 & 9 \end{pmatrix}$$

$$L' = \begin{pmatrix} 2 & 0 \\ 7 & 3 \end{pmatrix}, \quad U' = \begin{pmatrix} 1 & 5 \\ 0 & 3 \end{pmatrix}$$

What can we do with LU-dcomposition

- Solving linear systems
- Find inverse
$$A = LU, \quad A^{-1} = U^{-1}L^{-1}$$
- Compute determinant
$$\det(A) = \det(L)\det(U)$$

where $\det(L)$ and $\det(U$ are easy to compute - product of diagonal entries